is in effect given in Serret's Algebra, and the solution is wonderfully simple. Write $A, B, C = ac - b^2, ad - bc, bd - c^2$, $\lambda = \sqrt{-\tfrac{1}{3}}$, Ω the discriminant $a^2d^2 +$ &c., then the values are

$$\alpha = \frac{-\lambda\sqrt{\Omega} + B}{2\lambda\sqrt{\Omega}},\ \beta = \frac{2C}{2\lambda\sqrt{\Omega}},\ \gamma = \frac{-2A}{2\lambda\sqrt{\Omega}},\ \delta = \frac{-\lambda\sqrt{\Omega} - B}{2\lambda\sqrt{\Omega}},$$

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represent one and the same double tangent (in fact we get each of these equations, and so in all $22 + 6 + 6, = 34$, 6 double tangents too many, if the two were not identical), but the *a posteriori* verification of the identity of the two equations is not by any means easy.

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then the quartic surface $\sqrt{x\xi} + \sqrt{y\eta} + \sqrt{z\zeta} = 0$, has the 16 singular tangent planes,

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For instance, if $N = 10$, the arrangement is

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Of course for the exceptional numbers $N = 1, 2, 5, 7, 12, 15$, &c., there is just one partition which is neither I nor D; and according as it is O or E we have in the product a coefficient -1 or $+1$.

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I will now explain what the system of Nonions consists in and how I have been concerned with it.

The calculus of Quaternions, one of the greatest of all mathematical discoveries, is a certain system of algebra applied to geometry. A quaternion is a four-dimensional quantity; that is to say, its value cannot be precisely expressed without the use of a set of four numbers. It is much as if a geographical position should be expressed by a single algebraical letter; the value of this letter could only be defined by the use of two numbers, say the Latitude and Longitude. There are various ways in which a quaternion may be conceived to be measured and various different sets of four numbers by which its value may be defined. Of all these modes, Hamilton, the author of the algebra, selected one as the standard. Namely, he conceived the general quaternion q to be put into the form

$$q = xi + yj + zk + w,$$

where x, y, z, w, are four ordinary numbers, while i, j, k, are peculiar units, subject to singular laws of multiplication. For $ij = -ji$, the order of the factors being material, as shown in this multiplication table, where the first factor is entered at the side, the second at the top, and the product is found in the body of the table.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

As long as x, y, z, and w in Hamilton's standard tetranomial form are confined to being *real* numbers, as he usually supposed them to be, no simpler or more advantageous form of conceiving the measurement of a quaternion can be found. But my father, Benjamin Peirce, made the profound, original, and pregnant discovery that when x, y, z, w are permitted to be imaginaries, there is another very different and preferable system of measurement of a quaternion. Namely, he showed that the general quaternion, q, can be put into the form

$$q = xi + yj + zk + wl,$$

where x, y, z, w, are real or imaginary numbers, while i, j, k, l, are peculiar units whose multiplication obeys this table.

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

A quaternion does not cease to be a quaternion by being measured upon one system rather than another. Any quantity belonging to the algebra is a quaternion; the algebra itself is "quaternions." The usual formulae of the calculus have no reference to any tetranomial form, and such a form might even be dispensed with altogether.

While my father was making his investigations in multiple algebra I was, in my humble way, studying the logic of relatives and an algebraic notation for it; and in the ninth volume of the *Memoirs of the American Academy of Arts and Sciences*, appeared my first paper upon the subject. In this memoir, I was led, from logical considerations that are patent to those who read it, to endeavor to put the general expression of any linear associative algebra into a certain form; namely as a linear expression in certain units which I wrote thus:

$$\begin{array}{lll} (u_1 : u_1) & (u_1 : u_2) & (u_1 : u_3), \text{ etc.,} \\ (u_2 : u_1) & (u_2 : u_2) & (u_2 : u_3), \text{ etc.,} \\ (u_3 : u_1) & (u_3 : u_2) & (u_3 : u_3), \text{ etc.,} \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

These forms, in their multiplication, follow these rules:

$$(u_a : u_b)(u_b : u_c) = (u_a : u_c) \qquad (u_a : u_b)(u_c : u_d) = 0.$$

I said, "I can assert, upon reasonable inductive evidence, that all such algebras can be interpreted on the principles of the present notation in the same way," and consequently can be put into this form. I afterwards published a proof of this. I added that this amounted to saying that "all such algebras are complications and modifications of the . . . Hamilton's quaternions." What I meant by this appears plainly in the memoir. It is that any algebra that can be put into the form proposed by me is thereby referred to an algebra of a certain class (afterwards named *quadrates* by Professor Clifford) which present so close an analogy with quaternions that they may all be considered as mere complications of that algebra. Of these algebras, I gave as an example, the multiplication table of that one which Professor Clifford afterward named *nonions*.* This is the passage: "For example, if we have three classes of individuals, u_1, u_2, u_3, which are related to one another in pairs, we may put

$$\begin{array}{lll} u_1 : u_1 = i & u_1 : u_2 = j & u_1 : u_3 = k \\ u_2 : u_1 = l & u_2 : u_2 = m & u_2 : n_3 = n \\ u_3 : u_1 = o & u_3 : u_2 = p & u_3 : u_3 = q \end{array}$$

and by (155) we get the multiplication table

	i	j	k	l	m	n	o	p	q"
i	i	j	k	0	0	0	0	0	0
j	0	0	0	i	j	k	0	0	0
k	0	0	0	0	0	0	i	j	k
l	l	m	n	0	0	0	0	0	0
m	0	0	0	l	m	n	0	0	0
n	0	0	0	0	0	0	l	m	n
o	o	p	q	0	0	0	0	0	0
p	0	0	0	o	p	q	0	0	0
q	0	0	0	0	0	0	o	p	q

It will be seen that the system of nonions is not a group but an algebra; that just as the word "quaternion" is not restricted to the three perpendicular vectors and unity, so a nonion is any quantity of this nine-fold algebra.

So much was published by me in 1870; and it then occurred either to my father or to me (probably in conversing together) that since this algebra was thus shown (through his form of quaternions) to be the strict analogue of quaternions, there ought to be a form of it analogous to Hamilton's standard tetranomial form of quaternions. That form, either he or I certainly found. I cannot remember, after so many years, which first looked for it; whichever did must have found it at once. I cannot tell what his method of search would have been, but I can show what my own must have been. The following course of reasoning was so obtrusive that I could not have missed it.

Hamilton's form of quaternions presents a group of four square-roots of unity. Are there, then, in nonions, nine independent cube-roots of unity, forming a group? Now, unity upon my system of notation was written thus:

$$(u_1 : u_1) + (u_2 : u_2) + (u_3 : u_3).$$

Two independent cube-roots of this suggest themselves at once, they are

$$(u_1 : u_2) + (u_2 : u_3) + (u_3 : u_1)$$
$$(u_3 : u_2) + (u_2 : u_1) + (u_1 : u_3).$$

In fact these are hinted at in my memoir, p. 53. Then, it must have immediately occurred to me, from the most familiar properties of the imaginary roots of unity, that instead of the coefficients

$$1, \quad 1, \quad 1,$$

I might substitute

$$1, \quad g, \quad g^2,$$

or

$$1, \quad g^2, \quad g,$$

where g is an imaginary cube-root of unity. The nine cube-roots of unity

*It would have been more accurately analogical, perhaps, to call it *novenions*.

thus obtained are obviously independent and obviously form a group. Thus the problem is solved by a method applicable to any other quadrate.

My father, with his strong partiality for my performances, talked a good deal about the algebra of nonions in general and these forms in particular; and they became rather widely known as mine. Yet it is clear that the only real merit in the discovery lay in my father's transformation of quaternions. In 1875, when I was in Germany, my father wrote to me that he was going to print a miscellaneous paper on multiple algebra and he wished to have it accompanied by a paper by me, giving an account of what I had found out. I wrote such a paper, and sent it to him; but somehow all but the first few pages of the manuscript were lost, a circumstance I never discovered till I saw the part that had reached him (and which he took for the whole) in print. I did not afterward publish the matter, because I did not attach much importance to it, and because I thought that too much had been made, already, of the very simple things I had done.

I here close the narrative. The priority of publication of the particular group referred to belongs to Professor Sylvester. But most readers will agree that he could not have desired to print it without making any allusion to my work, and that to say the group could be derived from my algebra was not too much.

C. S. PEIRCE.

A NOTE ON THE WORD "SOPHY" IN SHAKESPEARE'S TWELTH NIGHT.

Act II, sc. 5, 166: *Fabian.* I will not give my part of this sport for a pension of thousands to be paid from the *Sophy.*

Act III, sc. 4, 265: *Sir Toby.* They say he has been fencer to the *Sophy.*

These passages are well illustrated in a letter of a contemporary of Shakespeare, the Italian traveller, Pietro Della Valle.* The letter is dated Ispahan, March 17, 1617, less than a year after Shakespeare's death, and goes into much detail about the origin of the word "Sofi," which was a dynastic title, and hence disappeared with the extinction of the dynasty (vol. i, p. 464 of the Brighton ed. of 1843). Especially interesting is what Pietro has to say about Sir Robert Shirley, whose adventures in Persia had made the Sophy so familiar a name in England. The shah, he says, always wears a red cap, "like the other *quizilbasci*, or Turkoman soldiers," on certain solemnities. This is called *tag* or crown, and is the sign of belonging to the military and the nobility. This *tag* is sometimes conferred on foreigners who take service with the king, "just as an order of knighthood with us," but this happens seldom, and a well informed person told Pietro that he had seen it conferred only once in fifteen years. The bestowal of the *tag* is accompanied with great ceremonies, the king putting his own *tag* on the head of the person who is to receive the honor. And now we will let Pietro tell the Shirley story in his own sour-sweet way:

"In questo modo fu dato il *tag* a quel don Roberto Serley inglese, che gli anni passati venne in Roma ambasciador di questo re a papa Paolo, e adesso torna un' altra volta a tutti i principi della cristianità; ed ho inteso qui che don Roberto lo domandò: ma io a dire il vero, non solo non domanderei giammai tal cosa al re di Persia, ma mi dispiacerebbe sopra modo quando egli me l'offerisse; perchè non so come un cristiano possa lecitamente portar quell' insegna, che, insieme con l'onorevole della militar nobiltà, ha congiunto anche in sè non poco del superstizioso della falsa loro setta; onde, per rimediare a ciò, conforme io penso, don Roberto intendo che in cristianità soleva portarvi in cima una croce. Ma oltre di questo, io non tengo che un Franco debba ambire di portare un' insegna d' onore, che è comune a molte migliaia di schiavi e di soldati ordinari; però per chi avesse voglia di vivere in Persia, come forse deve avere il detto don Roberto, potrebbe passar per cosa, se non desiderabile, almeno onorata."

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In the evening, there was a social assembly of the officers and students and their friends. The library and halls of the University were thrown open to a company of gentlemen and ladies, several hundred in number.

I will now explain what the system of Nonions consists in and how I have been concerned with it.

The calculus of Quaternions, one of the greatest of all mathematical discoveries, is a certain system of algebra applied to geometry. A quaternion is a four-dimensional quantity; that is to say, its value cannot be precisely expressed without the use of a set of four numbers. It is much as if a geographical position should be expressed by a single algebraical letter; the value of this letter could only be defined by the use of two numbers, say the Latitude and Longitude. There are various ways in which a quaternion may be conceived to be measured and various different sets of four numbers by which its value may be defined. Of all these modes, Hamilton, the author of the algebra, selected one as the standard. Namely, he conceived the general quaternion q to be put into the form

$$q = xi + yj + zk + w,$$

where x, y, z, w, are four ordinary numbers, while i, j, k, are peculiar units, subject to singular laws of multiplication. For $ij = -ji$, the order of the factors being material, as shown in this multiplication table, where the first factor is entered at the side, the second at the top, and the product is found in the body of the table.

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

As long as x, y, z, and w in Hamilton's standard tetranomial form are confined to being *real* numbers, as he usually supposed them to be, no simpler or more advantageous form of conceiving the measurement of a quaternion can be found. But my father, Benjamin Peirce, made the profound, original, and pregnant discovery that when x, y, z, w are permitted to be imaginaries, there is another very different and preferable system of measurement of a quaternion. Namely, he showed that the general quaternion, q, can be put into the form

$$q = xi + yj + zk + wl,$$

where x, y, z, w, are real or imaginary numbers, while i, j, k, l, are peculiar units whose multiplication obeys this table.

	i	j	k	l
i	i	j	0	0
j	0	0	i	j
k	k	l	0	0
l	0	0	k	l

A quaternion does not cease to be a quaternion by being measured upon one system rather than another. Any quantity belonging to the algebra is a quaternion; the algebra itself is "quaternions." The usual formulae of the calculus have no reference to any tetranomial form, and such a form might even be dispensed with altogether.

While my father was making his investigations in multiple algebra I was, in my humble way, studying the logic of relatives and an algebraic notation for it; and in the ninth volume of the *Memoirs of the American Academy of Arts and Sciences*, appeared my first paper upon the subject. In this memoir, I was led, from logical considerations that are patent to those who read it, to endeavor to put the general expression of any linear associative algebra into a certain form; namely as a linear expression in certain units which I wrote thus:

$$\begin{array}{lll} (u_1 : u_1) & (u_1 : u_2) & (u_1 : u_3), \text{ etc.,} \\ (u_2 : u_1) & (u_2 : u_2) & (u_2 : u_3), \text{ etc.,} \\ (u_3 : u_1) & (u_3 : u_3) & (u_3 : u_3), \text{ etc.,} \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

These forms, in their multiplication, follow these rules:

$$(u_a : u_b)(u_b : u_c) = (u_a : u_c) \qquad (u_a : u_b)(u_c : u_d) = 0.$$

I said, "I can assert, upon reasonable inductive evidence, that all such algebras can be interpreted on the principles of the present notation in the same way," and consequently can be put into this form. I afterwards published a proof of this. I added that this amounted to saying that "all such algebras are complications and modifications of the . . . Hamilton's quaternions." What I meant by this appears plainly in the memoir. It is that any algebra that can be put into the form proposed by me is thereby referred to an algebra of a certain class (afterwards named *quadrates* by Professor Clifford) which present so close an analogy with quaternions that they may all be considered as mere complications of that algebra. Of these algebras, I gave as an example, the multiplication table of that one which Professor Clifford afterward named *nonions*.* This is the passage: "For example, if we have three classes of individuals, u_1, u_2, u_3, which are related to one another in pairs, we may put

$$\begin{array}{lll} u_1 : u_1 = i & u_1 : u_2 = j & u_1 : u_3 = k \\ u_2 : u_1 = l & u_2 : u_2 = m & u_2 : u_3 = n \\ u_3 : u_1 = o & u_3 : u_2 = p & u_3 : u_3 = q \end{array}$$

and by (155) we get the multiplication table

	i	j	k	l	m	n	o	p	q
i	i	j	k	0	0	0	0	0	0
j	0	0	0	i	j	k	0	0	0
k	0	0	0	0	0	0	i	j	k
l	l	m	n	0	0	0	0	0	0
m	0	0	0	l	m	n	0	0	0
n	0	0	0	0	0	0	l	m	n
o	o	p	q	0	0	0	0	0	0
p	0	0	0	o	p	q	0	0	0
q	0	0	0	0	0	0	o	p	q

It will be seen that the system of nonions is not a group but an algebra; that just as the word "quaternion" is not restricted to the three perpendicular vectors and unity, so a nonion is any quantity of this nine-fold algebra.

So much was published by me in 1870; and it then occurred either to my father or to me (probably in conversing together) that since this algebra was thus shown (through his form of quaternions) to be the strict analogue of quaternions, there ought to be a form of it analogous to Hamilton's standard tetranomial form of quaternions. That form, either he or I certainly found. I cannot remember, after so many years, which first looked for it; whichever did must have found it at once. I cannot tell what his method of search would have been, but I can show what my own must have been. The following course of reasoning was so obtrusive that I could not have missed it.

Hamilton's form of quaternions presents a group of four square-roots of unity. Are there, then, in nonions, nine independent cube-roots of unity, forming a group? Now, unity upon my system of notation was written thus:

$$(u_1 : u_1) + (u_2 : u_2) + (u_3 : u_3).$$

Two independent cube-roots of this suggest themselves at once, they are

$$(u_1 : u_2) + (u_2 : u_3) + (u_3 : u_1)$$
$$(u_3 : u_2) + (u_2 : u_1) + (u_1 : u_3).$$

In fact these are hinted at in my memoir, p. 53. Then, it must have immediately occurred to me, from the most familiar properties of the imaginary roots of unity, that instead of the coefficients

$$1, \quad 1, \quad 1,$$

I might substitute

$$1, \quad g, \quad g^2,$$

or

$$1, \quad g^2, \quad g,$$

where g is an imaginary cube-root of unity. The nine cube-roots of unity

*It would have been more accurately analogical, perhaps, to call it *novenions*.

thus obtained are obviously independent and obviously form a group. Thus the problem is solved by a method applicable to any other quadrate.

My father, with his strong partiality for my performances, talked a good deal about the algebra of nonions in general and these forms in particular; and they became rather widely known as mine. Yet it is clear that the only real merit in the discovery lay in my father's transformation of quaternions. In 1875, when I was in Germany, my father wrote to me that he was going to print a miscellaneous paper on multiple algebra and he wished to have it accompanied by a paper by me, giving an account of what I had found out. I wrote such a paper, and sent it to him; but somehow all but the first few pages of the manuscript were lost, a circumstance I never discovered till I saw the part that had reached him (and which he took for the whole) in print. I did not afterward publish the matter, because I did not attach much importance to it, and because I thought that too much had been made, already, of the very simple things I had done.

I here close the narrative. The priority of publication of the particular group referred to belongs to Professor Sylvester. But most readers will agree that he could not have desired to print it without making any allusion to my work, and that to say the group could be derived from my algebra was not too much.

C. S. PEIRCE.

A NOTE ON THE WORD "SOPHY" IN SHAKESPEARE'S TWELFTH NIGHT.

Act II, sc. 5, 166: *Fabian.* I will not give my part of this sport for a pension of thousands to be paid from the *Sophy*.

Act III, sc. 4, 265: *Sir Toby.* They say he has been fencer to the *Sophy*.

These passages are well illustrated in a letter of a contemporary of Shakespeare, the Italian traveller, Pietro Della Valle.* The letter is dated Ispahan, March 17, 1617, less than a year after Shakespeare's death, and goes into much detail about the origin of the word "Sofi," which was a dynastic title, and hence disappeared with the extinction of the dynasty (vol. i, p. 464 of the Brighton ed. of 1843). Especially interesting is what Pietro has to say about Sir Robert Shirley, whose adventures in Persia had made the Sophy so familiar a name in England. The shah, he says, always wears a red cap, "like the other *quizilbasci*, or Turkoman soldiers," on certain solemnities. This is called *tag* or crown, and is the sign of belonging to the military and the nobility. This *tag* is sometimes conferred on foreigners who take service with the king, "just as an order of knighthood with us," but this happens seldom, and a well informed person told Pietro that he had seen it conferred only once in fifteen years. The bestowal of the *tag* is accompanied with great ceremonies, the king putting his own *tag* on the head of the person who is to receive the honor. And now we will let Pietro tell the Shirley story in his own sour-sweet way:

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