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If ϕu denotes any elliptic function of the r^{th} degree with the periods 2ω, $2\omega'$, and if σu has the same pair of periods, then we can always determine the $2r+1$ quantities

$$u_1, u_2, \ldots u_r;\ v_1, v_2, \ldots v_r,\ C$$

so that

$$\phi(u) = C.\frac{\sigma(u-u_1)\,\sigma(u-u_2)\ldots\sigma(u-u_r)}{\sigma(u-v_1)\,\sigma(u-v_2)\ldots\sigma(u-v_r)};$$

which proposition is capable of inversion. An analogous theorem in regard to $\wp u$ is, if

$$u_0, u_1, u_2, \ldots u_n$$

denote $n+1$ independent variables, then the function

$$\phi(u_0, u_1, u_2, \ldots u_n) = \begin{vmatrix} 1 & \wp u_0 & \wp' u_0 \ldots \wp^{(n-1)} u_0 \\ 1 & \wp u_1 & \wp' u_1 \ldots \wp^{(n-1)} u_1 \\ 1 & \wp u_2 & \wp' u_2 \ldots \wp^{(n-1)} u_2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 1 & \wp u_n & \wp' u_n \ldots \wp^{(n-1)} u_n \end{vmatrix}$$

is an elliptic function of the degree $n+1$ of any one of the arguments $u_0, u_1 \ldots u_n$. In general "every unique elliptic function $\phi(u)$ is expressible as a rational function of $\wp u$ and the first derivative $\wp' u$ with the same pair of periods 2ω, $2\omega'$ as $\phi(u)$; and in like manner $\wp u$ and $\wp' u$ are expressible as rational functions of ϕu and $\phi' u$".

With the function σu are closely connected the following

$$\sigma_1 u = \frac{e^{-\eta u}\sigma(\omega+u)}{\sigma\omega} = \frac{e^{\eta u}\sigma(\omega-u)}{\sigma\omega}$$

$$\sigma_2 u = \frac{e^{-\eta'' u}\sigma(\omega''+u)}{\sigma\omega''} = \frac{e^{\eta'' u}\sigma(\omega''-u)}{\sigma\omega''}$$

$$\sigma_3 u = \frac{e^{-\eta' u}\sigma(\omega'+u)}{\sigma\omega'} = \frac{e^{\eta' u}\sigma(\omega'-u)}{\sigma\omega'}$$

where ω, ω' are the half periods, and $\omega+\omega'=\omega''$, $\frac{\sigma'\omega}{\sigma\omega}=\eta$, $\frac{\sigma'\omega'}{\sigma\omega'}=\eta'$, $\eta+\eta'=\eta''$. By inserting in the "pocket edition" for v the values respectively ω, ω'', ω', we have

$$\wp u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2,\quad \wp u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2,\quad \wp u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2$$

whereby the following relations are established for the differences of the roots. Remembering that $\wp\omega = e_1$, $\wp\omega'' = e_2$, $\wp\omega' = e_3$.

$$\sqrt{e_1-e_2} = \frac{\sigma_2\omega}{\sigma\omega},\quad \sqrt{e_2-e_3} = \frac{\sigma_3\omega''}{\sigma\omega''},\quad \sqrt{e_1-e_3} = \frac{\sigma_3\omega}{\sigma\omega}$$

$$\sqrt{e_2-e_1} = \frac{\sigma_1\omega''}{\sigma\omega''},\quad \sqrt{e_3-e_2} = \frac{\sigma_2\omega'}{\sigma\omega'};\quad \sqrt{e_3-e_1} = \frac{\sigma_1\omega'}{\sigma\omega'}$$

where we assume $e_1 > e_2 > e_3$. If now we assume $R\left(\frac{\omega'}{\omega i}\right) > 0$, that is, the real component of the complex $\frac{\omega'}{\omega\sqrt{-1}} > 0$, so that in the geometrical representation

the point ω' lies "above" the right line joining $u=0$ and $u=\omega$, then

$$\sqrt{e_3-e_2} = -i\sqrt{e_2-e_3};\quad \sqrt{e_3-e_1} = -i\sqrt{e_1-e_3};\quad \sqrt{e_2-e_1} = -i\sqrt{e_1-e_2}.$$

If now we denote for convenience by λ, μ, ν the indices 1, 2, 3, and write

$$\frac{\sigma u}{\sigma_\lambda u} = \xi_{o\lambda},\quad \frac{\sigma_\mu u}{\sigma_\nu u} = \xi_{\mu\nu},\quad \frac{\sigma_\lambda u}{\sigma u} = \xi_{\lambda o},\ \text{etc.}$$

remembering that

$$\wp' u = -2\frac{\sigma_\lambda u.\sigma_\mu u.\sigma_\nu u}{\sigma u.\sigma u.\sigma u},$$

we easily obtain

$$\frac{d\xi_{o\lambda}}{du} = \xi_{\mu\lambda}\xi_{\nu\lambda},\quad \frac{d\xi_{\mu\nu}}{du} = -(e_\mu - e_\nu)\,\xi_{\lambda o}\xi_{o\nu},\quad \frac{d\xi_{\lambda o}}{du} = -\xi_{\mu o}\xi_{\nu o},$$

whence

$$\left(\frac{d\xi_{o\lambda}}{du}\right)^2 = \left(1-(e_\mu-e_\lambda)\xi^2_{o\lambda}\right)\left(1-(e_\nu-e_\lambda)\xi^2_{o\lambda}\right),$$

$$\left(\frac{d\xi_{\mu\nu}}{du}\right)^2 = (1-\xi^2_{\mu\nu})\left(e_\mu-e_\lambda+(e_\lambda-e_\nu)\xi^2_{\mu\nu}\right),$$

$$\left(\frac{d\xi_{\lambda o}}{du}\right)^2 = (\xi^2_{\lambda o}+e_\lambda-e_\mu)(\xi^2_{\lambda o}+e_\lambda-e_\nu),$$

and the four functions

$$\frac{\sigma u}{\sigma_\lambda u},\ \frac{1}{\sqrt{e_\mu-e_\lambda}}\frac{\sigma_\mu u}{\sigma_\nu u},\ \frac{1}{\sqrt{e_\nu-e_\lambda}}\frac{\sigma_\nu u}{\sigma_\mu u},\ \frac{1}{\sqrt{e_\mu-e_\lambda}\sqrt{e_\nu-e_\lambda}}\frac{\sigma_\lambda u}{\sigma u}$$

satisfy the same differential equation

$$\left(\frac{d\xi}{du}\right)^2 = \left(1-(e_\mu-e_\lambda)\xi_2\right)\left(1-(e_\nu-e_\lambda)\xi^2\right).$$

But the English reader will desire to know in what connection the system of Weierstrass stands to the more widely known systems of Jacobi and Legendre. If we define the k of Jacobi by the equation

$$k^2 = \frac{e_2-e_3}{e_1-e_3},$$

then the following relations are established between the sigma-quotients and Jacobi's functions. We give only three as specimens, replacing the λ, μ, ν by 1, 2, 3.

$$\frac{\sigma u}{\sigma_3 u} = \frac{1}{\sqrt{e_1-e_3}}\,\mathrm{sn}(\sqrt{e_1-e_3}.u,\ k)$$

$$\frac{\sigma_1 u}{\sigma_3 u} = \mathrm{cn}(\sqrt{e_1-e_3}.u,\ k)$$

$$\frac{\sigma_2 u}{\sigma_3 u} = \mathrm{dn}(\sqrt{e_1-e_3}.u,\ k).$$

Not all the sigma-quotients are so nearly identical with Jacobi's functions, but in all cases the argument u appears multiplied with the same factor $\sqrt{e_1-e_3}$ which is the largest of the three root-differences.

In the defining equation $k^2 = \frac{e_2-e_3}{e_1-e_3}$ and the corresponding one $k'^2 = \frac{e_1-e_2}{e_1-e_3}$ both of these quantities if real must be greater than zero and less than unity.

They will be real if the points in the plane representing e_1, e_2, e_3 lie in the same straight line, when mod. e_2 must be intermediate between mod. e_1 and mod. e_3 in magnitude. Then if we understand by K and K' the simplest values of the integrals

$$\int_0^1 \frac{dx}{\sqrt{1-x^2 . 1-k^2x^2}}; \quad \int_0^1 \frac{dx}{\sqrt{1-x^2 . 1-k'^2x^2}}$$

respectively, taking those values of the radicals whose real components are positive, we shall have

$$\omega_1 \sqrt{e_1 - e_3} = K, \quad \omega_3 \sqrt{e_1 - e_3} = iK',$$
$$\omega_2 = \omega_1 + \omega_3,$$

and $2\omega_1$, $2\omega_3$ are the primitive pair of periods for the before mentioned $\wp u$, so that as above
$$\wp\omega_1 = e_1, \quad \wp\omega_2 = e_2, \quad \wp\omega_3 = e_3.$$

It ought to be mentioned that $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ can also be defined in the same simple manner as σu by means of infinite products. If we write

$$w_1 = (2\mu + 1)\omega + 2\mu'\omega', \qquad w_2 = (2\mu + 1)\omega + (2\mu' + 1)\omega'$$
$$w_3 = 2\mu\omega + (2\mu' + 1)\omega', \qquad [\mu, \mu' = 0, \pm 1, \pm 2 \ldots \pm \infty]$$

then in general, for $\lambda = 1, 2, 3$,

$$\sigma_\lambda u = e^{-\frac{1}{2}e_\lambda u^2} \Pi_{w_\lambda} \left(1 - \frac{u}{w_\lambda}\right) e^{\frac{u}{w_\lambda} + \frac{1}{2}\frac{u^2}{w_\lambda^2}}.$$

Finally to show the relation in which the sigma functions stand to the ϑ-functions of Jacobi, we find

$$\sigma u = \frac{2\omega}{\pi} e^{2\eta\omega v^2} . \frac{2h^{\frac{1}{4}} \sin v\pi - 2h^{\frac{9}{4}} \sin 3v\pi + 2h^{\frac{25}{4}} \sin 5v\pi - \ldots}{2h^{\frac{1}{4}} \quad - 3.2.h^{\frac{9}{4}} \quad + 5.2h^{\frac{25}{4}} \quad - \ldots} = 2\omega e^{2\eta\omega v^2} . \frac{\vartheta_0(v)}{\vartheta_0'(o)}$$

$$\sigma_1 u = e^{2\eta\omega v^2} . \frac{2h^{\frac{1}{4}} \cos v\pi + 2h^{\frac{9}{4}} \cos 3v\pi + 2h^{\frac{25}{4}} \cos 5v\pi + \ldots}{2h^{\frac{1}{4}} \quad + 2h^{\frac{9}{4}} \quad + 2h^{\frac{25}{4}} \quad + \ldots} = e^{2\eta\omega v^2} . \frac{\vartheta_1(v)}{\vartheta_1(o)}$$

$$\sigma_2 u = e^{2\eta\omega v^2} . \frac{1 + 2h \cos 2v\pi + 2h^4 \cos 4v\pi + 2h^9 \cos 6v\pi + \ldots}{1 + 2h \quad + 2h^4 \quad + 2h^9 \quad + \ldots} = e^{2\eta\omega v^2} . \frac{\vartheta_2(v)}{\vartheta_2(o)}$$

$$\sigma_3 u = e^{2\eta\omega v^2} . \frac{1 - 2h \cos 2v\pi + 2h^4 \cos 4v\pi - 2h^9 \cos 6v\pi + \ldots}{1 - 2h \quad + 2h^4 \quad - 2h^9 \quad + \ldots} = e^{2\eta\omega v^2} . \frac{\vartheta_3(v)}{\vartheta_3(o)}$$

where $h = e^{\frac{\omega'}{\omega}\pi i}$, $v = \frac{u}{2\omega}$, $\eta = \frac{\sigma'\omega}{\sigma\omega}$.

The functions $\vartheta_0(v)$, $\vartheta_1(v)$, $\vartheta_2(v)$, $\vartheta_3(v)$ as here employed coincide respectively with Jacobi's $\vartheta_1(xq)$, $\vartheta_2(xq)$, $\vartheta_3(xq)$, $\vartheta(xq)$, if we write $v\pi = x$ and $h = q$.

But anything more than a slight account of Weierstrass' system, showing in particular its main points of contact with Jacobi's, would be beyond the intention of this paper. It is to be hoped that Weierstrass' ideas in the function-theory will soon find that widespread recognition which they undoubtedly merit. In a future paper I hope to exhibit the system in greater detail, in particular the formulæ of transformation, showing their analogies to the formulæ of Jacobi.

On Quadruple Theta-Functions.

BY THOMAS CRAIG, *Johns Hopkins University.*

PART II.

In the following I employ the notation used by Schottky in his "*Abriss einer Theorie der Abelschen Functionen von drei Variabeln.*" On page 18, Schottky gives the fundamental theorem bearing upon his particular notation; it is as follows: "*Es ist möglich, ein System primitiver Indices*

$$1, 2, 3 \ldots 2\rho + 1$$

und einen ausgezeichneten ε so zu wählen, dass εa ein grader Index ist, wenn die Anzahl der primitiven Indices, aus denen a zusammengesetzt ist $\equiv \rho$ oder $\rho + 1$ mod. 4 ist, dagegen ein ungrader, wenn diese Anzahl $\equiv \rho + 2$ oder $\rho - 1$ mod. 4 ist."

For $\rho = 4$ the "primitive indices" are nine in number, and they may in general be denoted by the letters

$$k, l, m, n, p, q, r, s, t.$$

All of the characteristics of the quadruple theta-functions, with the exception of (0), may be represented by certain combinations of these letters, viz. by taking them one at a time, two at a time, three at a time and four at a time. We have thus:

	Number of cases.
The index (0)	1
The primitive indices taking one at a time	9
" " " two "	36
" " " three "	84
" " " four "	126
	256

The even functions, 136 in number, are given by the first, second and fifth of these cases, and the odd functions, 120 in number, by the third and fourth cases. That is, the even functions will have the suffixes o, k and $klmn$, and the odd functions will have the suffixes kl and klm. In the numbers of the *Annales de* *l'École Normale* for June, July and August, 1883, M. Brunel has investigated the relations similar to the Göpel and Kummer relations for the double theta-functions which exist in the case of the triple theta-functions. I propose in what follows to employ Brunel's method in working out the corresponding relations connecting the quadruple functions. Brunel starts out from certain relations given by Schottky in the *Nachtrag* to the above mentioned book "*Ueber die hyperelliptischen Functionen dreier Variabeln*," and uses a method which is fundamentally the same as that employed by Brioschi in his paper already referred to in Part I of this article,* but the manner in which he develops it is simpler than would be possible had he employed, without alteration, the method indicated by Brioschi.

I shall use almost without change the notation employed by Brunel, only altering it when the greater complexity of the present case makes it desirable. Following Schottky, write first:

$$\frac{L_k^4}{L_0^4}=\frac{-1}{(a_l-a_k)(a_m-a_k)(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)}$$

$$\frac{L_{kl}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_m-a_k)(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_m-a_l)(a_n-a_l)(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\end{array}\right.}$$

$$\frac{L_{klm}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_n-a_l)(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\\ \times(a_n-a_m)(a_p-a_m)(a_q-a_m)(a_r-a_m)(a_s-a_m)(a_t-a_m)\end{array}\right.}$$

$$\frac{L_{klmn}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\\ \times(a_p-a_m)(a_q-a_m)(a_r-a_m)(a_s-a_m)(a_t-a_m)\\ \times(a_p-a_n)(a_q-a_n)(a_r-a_n)(a_s-a_n)(a_t-a_n)\end{array}\right.}$$

Now consider the functions P defined by the equations

1. $$\frac{L_0}{L_0}P_0=\frac{\theta_0}{\theta_0},\quad \frac{L_k}{L_0}P_k=\frac{\theta_k}{\theta_0},\quad \frac{L_{kl}}{L_0}P_{kl}=\frac{\theta_{kl}}{\theta_0},\quad \frac{L_{klm}}{L_0}P_{klm}=\frac{\theta_{klm}}{\theta_0},\quad \frac{L_{klmn}}{L_0}P_{klmn}=\frac{\theta_{klmn}}{\theta_0},$$

* Brioschi.—*La relazione di Göpel per funzioni iperellittiche d'ordine qualunque.* Annali di Matematica, Serie II^a, Tomo X°.

then, following Weierstrass and Schottky, and writing

2. $$R(x)=(a_k-x)(a_l-x)(a_m-x)(a_n-x)(a_p-x)(a_q-x)(a_r-x)(a_s-x)(a_t-x)$$
$$\phi(x)=(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

we have

$$P_0=1 \quad \ldots\ldots \quad 1 \text{ function } P_0$$
$$P_k=\sqrt{(a_k-x_1)(a_k-x_2)(a_k-x_3)(a_k-x_4)} \quad \ldots\ldots \quad 9 \text{ functions } P_k$$

3. $$P_{kl}=P_kP_l\sum_{i=1}^{i=4}\frac{\sqrt{R(x_i)}}{(a_k-x_i)(a_l-x_i)\,\varphi'(x_i)} \quad \ldots\ldots \quad 36 \text{ functions } P_{kl}$$

$$P_{klm}=P_kP_lP_m\sum_{i=1}^{i=4}\frac{\sqrt{R(x_i)}}{(a_k-x_i)(a_l-x_i)(a_m-x_i)\,\varphi'(x_i)} \quad \ldots\ldots \quad 84 \text{ functions } P_{klm}$$

$$P_{klmn}=P_kP_lP_mP_n\sum_{i=1}^{i=4}\frac{\sqrt{R(x_i)}}{(a_k-x_i)(a_l-x_i)(a_m-x_i)(a_n-x_i)\,\varphi'(x_i)} \ldots 126 \text{ functions } P_{klmn}$$

making in all 256 P-functions replacing the 256 Θ-functions. In these equations the letters k, l, m and n are all different from each other. We have now to determine the linear relations existing between the squares of these P-functions and those existing between their products taken two and two. Write

4. $$\Sigma x_i=\alpha,\quad \Sigma x_ix_j=\beta,\quad \Sigma x_ix_jx_k=\gamma,\quad x_1x_2x_3x_4=\delta.$$

The summations to be taken from 1 to 4 and i, j, k all having different values. Further write

$$(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4)=-\theta,$$

that is

5. $$\begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3\\ x_1^2 & x_2^2 & x_3^2 & x_4^2\\ x_1 & x_2 & x_3 & x_4\\ 1 & 1 & 1 & 1\end{vmatrix}=-\theta;$$

and in general write

$$\begin{vmatrix} y_1^{n-1} & y_2^{n-1} & \ldots & y_n^{n-1}\\ y_1^{n-2} & y_2^{n-2} & \ldots & y_n^{n-2}\\ \ldots & \ldots & \ldots & \ldots\\ y_1 & y_2 & \ldots & y_n\\ 1 & 1 & \ldots & 1\end{vmatrix}\equiv|y_1y_2\ldots y_n|,$$

that is

6. $$|x_1x_2x_3x_4|=-\theta.$$

The P-functions can now be written in the following manner,

$$P_0 = 1, \quad P_k = \sqrt{|a_k x_1||a_k x_2||a_k x_3||a_k x_4|}$$

$$P_{kl} = \frac{1}{|x_1 x_2 x_3 x_4|}\Big\{ |x_2 x_3 x_4| \sqrt{R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|}} - |x_3 x_4 x_1|\sqrt{} + |x_4 x_1 x_2|\sqrt{} - |x_1 x_2 x_3|\sqrt{} \Big\}$$

7. $$P_{klm} = \frac{1}{|x_1 x_2 x_3 x_4|}\Big\{ |x_2 x_3 x_4| \sqrt{R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4|}{|a_k x_1||a_l x_1||a_m x_1|}} - |x_3 x_4 x_1|\sqrt{} + |x_4 x_1 x_2|\sqrt{} - |x_1 x_2 x_3|\sqrt{} \Big\}$$

$$P_{klmn} = \frac{1}{|x_1 x_2 x_3 x_4|}\Big\{ |x_2 x_3 x_4| \sqrt{R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4||a_n x_2||a_n x_3||a_n x_4|}{|a_k x_1||a_l x_1||a_m x_1||a_n x_1|}} - |x_3 x_4 x_1|\sqrt{} + |x_4 x_1 x_2|\sqrt{} - |x_1 x_2 x_3|\sqrt{} \Big\}$$

It is of course perfectly obvious how to fill up the empty radical signs. The above forms for the P-functions are retained for the same reason that Brunel gives in the case of the triple theta-functions, that is, although the denominators under the radical signs are actually factors of $R(x_1)$, $R(x_2)$, $R(x_3)$ and $R(x_4)$, it is more convenient in the following transformations to retain the fractional form.

LINEAR RELATIONS BETWEEN THE SQUARES OF THE P-FUNCTIONS.

Take first the case of the functions with a single index, *i. e.* P_k, P_l, etc. We have

8. $$P_k^2 = |a_k x_1||a_k x_2||a_k x_3||a_k x_4|;$$

expanding this and using the notation given above, we have

9. $$P_k^2 = a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta.$$

As this is linear in the quantities α, β, γ, δ, and as k is any one of the primitive indices, we can, by assuming any five such relations, eliminate α, β, γ and δ; the result of the elimination is obviously

10. $$\begin{vmatrix} P_k^2 - a_k^4, & P_l^2 - a_l^4, & P_m^2 - a_m^4, & P_n^2 - a_n^4, & P_p^2 - a_p^4 \\ a_k^3 & a_l^3 & a_m^3 & a_n^3 & a_p^3 \\ a_k^2 & a_l^2 & a_m^2 & a_n^2 & a_p^2 \\ a_k & a_l & a_m & a_n & a_p \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or expanding this we have

11. $$P_k^2 |a_l a_m a_n a_p| + P_l^2 |a_m a_n a_p a_k| + P_m^2 |a_n a_p a_k a_l| + P_n^2 |a_p a_k a_l a_m| + P_p^2 |a_k a_l a_m a_n| = |a_k a_l a_m a_n a_p| P_0^2$$

Since $P_0 = 1$, this factor may be introduced solely for the sake of symmetry. If instead of eliminating α, β, γ and δ between five relations of the form (9) we eliminate α, β, γ, δ and 1 between six such relations, we have obviously

12. $$\begin{vmatrix} P_k^2 & P_l^2 & P_m^2 & P_n^2 & P_p^2 & P_q^2 \\ a_k^4 & a_l^4 & a_m^4 & a_n^4 & a_p^4 & a_q^4 \\ a_k^3 & a_l^3 & a_m^3 & a_n^3 & a_p^3 & a_q^3 \\ a_k^2 & a_l^2 & a_m^2 & a_n^2 & a_p^2 & a_q^2 \\ a_k & a_l & a_m & a_n & a_p & a_q \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or expanding

13. $$P_k^2 |a_l a_m a_n a_p a_q| - P_l^2 |a_m a_n a_p a_q a_k| + P_m^2 |a_n a_p a_q a_k a_l| - P_n^2 |a_p a_q a_k a_l a_m| + P_p^2 |a_q a_k a_l a_m a_n| - P_q^2 |a_k a_l a_m a_n a_p| = 0.$$

We have thus found the linear relations existing between the squares of the P-functions possessing a single suffix, or index, *i. e.* between the functions whose indices are $0\ k\ l\ m\ n\ p\ q\ r\ s\ t$, and it is seen that these functions form a group of ten such that any five being given the square of any one of the remaining five can be expressed as a linear function of the squares of the chosen five. Following Brunel I shall call this the group 0.

Consider next the case of the P-functions with two suffixes: for the square of any one of them, say P_{kl}, we have

14. $$\begin{aligned} P_{kl}^2 = \frac{1}{|x_1 x_2 x_3 x_4|^2}\Big\{ & |x_2 x_3 x_4|^2 . R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|} + \ldots \\ & - 2|x_2 x_3 x_4||x_3 x_4 x_1| \sqrt{R(x_1) R(x_2)}\, |a_k x_3||a_k x_4||a_l x_3||a_l x_4| \\ & + 2|x_2 x_3 x_4||x_4 x_1 x_2| \sqrt{R(x_1) R(x_3)}\, |a_k x_4||a_k x_2||a_l x_4||a_l x_2| \\ & - 2|x_2 x_3 x_4||x_1 x_2 x_3| \sqrt{R(x_1) R(x_4)}\, |a_k x_2||a_k x_3||a_l x_2||a_l x_3| \\ & - 2|x_3 x_4 x_1||x_4 x_1 x_2| \sqrt{R(x_2) R(x_3)}\, |a_k x_1||a_k x_4||a_l x_1||a_l x_4| \\ & + 2|x_3 x_4 x_1||x_1 x_2 x_3| \sqrt{R(x_2) R(x_4)}\, |a_k x_1||a_k x_3||a_l x_1||a_l x_3| \\ & - 2|x_4 x_1 x_2||x_1 x_2 x_3| \sqrt{R(x_3) R(x_4)}\, |a_k x_1||a_k x_2||a_l x_1||a_l x_2| \Big\}. \end{aligned}$$

It is possible to find a linear relation between four of these P-functions with two suffixes which is entirely rational, that is, a relation which shall not contain any of the quantities $\sqrt{R(x_i) R(x_j)}$. Take four of the functions P_{ki} which have the first suffix k in common, say P_{kl}, P_{km}, P_{kn}, P_{kp}, then in order that the radicals $\sqrt{R(x_i) R(x_j)}$ may disappear we must find a series of multipliers A, B, C, D, satisfying the equation

15. $$A|a_l x_3||a_l x_4| - B|a_m x_3||a_m x_4| + C|a_n x_3||a_n x_4| - D|a_p x_3||a_p x_4| = 0.$$

Giving A, B, C and D the following values,

16. $$A=|a_ma_na_p|, \quad B=|a_na_pa_l|$$
$$C=|a_pa_la_m|, \quad D=|a_la_ma_n|$$

it is easy to see that equation 15 is satisfied. Assuming then four equations of the same form as 14, we have

$$|a_ma_na_p|P^2_{kl}-|a_na_pa_l|P^2_{km}+|a_pa_la_m|P^2_{kn}-|a_la_ma_n|P^2_{kp}=$$
$$\frac{1}{|x_1x_2x_3x_4|^2}\Big[|a_ma_na_p|\left\{|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|}{|a_kx_1||a_lx_1|}+\dots\right\}$$

17. $$-|a_na_pa_l|\left\{|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_mx_1|}+\dots\right\}$$
$$+|a_pa_la_m|\left\{|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_nx_2||a_nx_3||a_nx_4|}{|a_kx_1||a_nx_1|}+\dots\right\}$$
$$-|a_la_ma_n|\left\{|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_px_2||a_px_3||a_px_4|}{|a_kx_1||a_px_1|}+\dots\right\}\Big].$$

Introducing the values of $R(x_1)$, $R(x_2)$, etc., it is not difficult to see that the first line of this equation may be thrown into the following form,

$$\frac{1}{|x_1x_2x_3x_4|^2}\{|a_ma_na_p||a_mx_1||a_nx_1||a_px_1|\,[\,|x_2x_3x_4||a_qx_1||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4|\,]\,\mathfrak{A}$$
$$-|a_na_pa_l||a_nx_1||a_px_1||a_lx_1|\,[\;\text{" " " " " " " "}\;]\,\mathfrak{B}$$
$$+|a_pa_la_m||a_px_1||a_lx_1||a_mx_1|\,[\;\text{" " " " " " " "}\;]\,\mathfrak{C}$$
$$-|a_la_ma_n||a_lx_1||a_mx_1||a_nx_1|\,[\;\text{" " " " " " " "}\;]\,\mathfrak{D}$$

where $\mathfrak{A}=|x_2x_3x_4||a_lx_2||a_lx_3||a_lx_4|-|x_3x_4x_1||a_lx_3||a_lx_4||a_lx_1|$
$+|x_4x_1x_2||a_lx_4||a_lx_1||a_lx_2|-|x_1x_2x_3||a_lx_1||a_lx_2||a_lx_3|$

and $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ are obtained by changing l into m, n and p respectively. We have then $$\mathfrak{A}=\mathfrak{B}=\mathfrak{C}=\mathfrak{D}=|x_1x_2x_3x_4|=\text{ say }\Delta.$$

Write for convenience

$$|x_2x_3x_4||a_qx_1||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4|=\Gamma_1,$$

then this becomes

$$\frac{\Delta\Gamma_1}{|x_1x_2x_3x_4|^2}\{|a_ma_na_p||a_mx_1||a_nx_1||a_px_1|-|a_na_pa_l||a_nx_1||a_px_1||a_lx_1|$$
$$+|a_pa_la_m||a_px_1||a_lx_1||a_mx_1|-|a_la_ma_n||a_lx_1||a_mx_1||a_nx_1|\}.$$

The term in the $\{\}$ is easily seen to be equal to $|a_la_ma_na_p|$. Equation 17 thus takes the form,

$$|a_ma_na_p|P^2_{kl}-|a_na_pa_l|P^2_{km}+|a_pa_la_m|P^2_{kn}-|a_la_ma_n|P^2_{kp}=$$

18. $$\frac{|a_la_ma_na_p|}{|x_1x_2x_3x_4|}[\Gamma_1-\Gamma_2+\Gamma_3-\Gamma_4].$$

Remembering to pay attention to the signs, we may write the term in [ ] as $\Sigma\Gamma$; then writing

19. $$\Sigma a_q=\lambda,\ \Sigma a_qa_r=\mu,\ \Sigma a_qa_ra_s=\nu,\ \Sigma a_qa_ra_sa_t=\pi,$$

we have

20. $$\Sigma\Gamma=\Sigma|x_2x_3x_4|(\pi-x_1\nu+x_1^2\mu-x_1^3\lambda+x_1^4)(a_k^3-a_k^2(x_2+x_3+x_4)$$
$$+a_k(x_2x_3+x_4x_2+x_3x_4)-x_2x_3x_4).$$

Now writing as above $\theta=-|x_1x_2x_3x_4|$

and introducing the abbreviations α, β, γ, δ and λ, μ, ν, π, it is not difficult to see that

21. $$\theta=|x_1x_2x_3x_4|\{a_k^4-a_k^3\alpha+a_k^2\beta-a_k\gamma+\delta-a_k^4+a_k^3\lambda-a_k^2\mu+a_k\nu-\pi\};$$

referring now to equation 9 we have

22. $$\theta=|x_1x_2x_3x_4|\{P_k^2-|a_ka_p||a_ka_q||a_ka_r||a_ka_s||a_ka_t|P_0^2\}.$$

Expanding equation 18 it becomes

$$a_k^3\{\pi\Sigma|x_1x_2x_3|-\nu\Sigma x_1|x_2x_3x_4|+\mu\Sigma x_1^2|x_2x_3x_4|-\lambda\Sigma x_1^3|x_2x_3x_4|+\Sigma x_1^4|x_2x_3x_4|\}$$
$$-a_k^2\{\pi\Sigma(x_2+x_3+x_4)|x_2x_3x_4|-\nu\Sigma x_1(x_2+x_3+x_4)|x_2x_3x_4|+\mu\Sigma x_1^2(x_2+x_3+x_4)|x_2x_3x_4|$$
$$-\lambda\Sigma x_1^3(x_2+x_3+x_4)|x_2x_3x_4|+\Sigma x_1^4(x_2+x_3+x_4)|x_2x_3x_4|\}$$
$$+a_k\{\pi\Sigma(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4|-\nu\Sigma x_1(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4|$$
$$+\mu\Sigma x_1^2(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4|-\lambda\Sigma x_1^3(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4|$$
$$+\Sigma x_1^4(x_2x_3+x_4x_2+x_3x_4)|x_2x_3x_4|\}-\{\pi\Sigma x_2x_3x_4|x_2x_3x_4|$$
$$-\nu x_1x_2x_3x_4\Sigma|x_2x_3x_4|+\mu x_1x_2x_3x_4\Sigma x_1|x_2x_3x_4|-\lambda x_1x_2x_3x_4\Sigma x_1^2|x_2x_3x_4|$$
$$+x_1x_2x_3x_4\Sigma x_1^3|x_2x_3x_4|\}=$$
$$|a_ma_na_p|P^2_{kl}-|a_na_pa_l|P^2_{km}+|a_na_la_m|P^2_{kn}-|a_la_ma_n|P^2_{kp}.$$

Of course in all these summations particular care must be taken to give the right signs to each term; for example, $\Sigma|x_2x_3x_4|$ means

$$|x_2x_3x_4|-|x_3x_4x_1|+|x_4x_1x_2|-|x_1x_2x_3|.$$

Using now equations 19 to 21 inclusive, we have, after simple reductions, for the reduced form of equation 18,

23. $$|a_ma_na_p|P^2_{kl}-|a_na_pa_l|P^2_{km}+|a_pa_la_m|P^2_{kn}-|a_la_ma_n|P^2_{kp}=$$
$$|a_la_ma_na_p|\{P_k^2-|a_ka_p||a_ka_q||a_ka_r||a_ka_s||a_ka_t|P_0^2\}.$$

The factor $P_0^2=1$ being introduced simply for the sake of symmetry.

Now advance all the letters after k, that is, change l, m, n, p, q, r, s, t into m, n, p, q, r, s, t, l, and 23 becomes

24. $$|a_na_pa_q|P^2_{km}-|a_pa_qa_m|P^2_{kn}+|a_qa_ma_n|P^2_{kp}-|a_ma_na_p|P^2_{kq}=$$
$$|a_ma_na_pa_q|\{P_k^2-|a_ka_q||a_ka_r||a_ka_s||a_ka_t||a_ka_l|P_0^2\}.$$

The coefficients of P_0^2 in 23 and 24 are respectively

$$\frac{[(a_m-a_n)(a_m-a_p)(a_n-a_p)(a_k-a_q)(a_k-a_r)(a_k-a_s)(a_k-a_t)]}{(a_l-a_m)(a_l-a_n)(a_l-a_p)(a_k-a_p)}$$

and

$$\frac{[(a_m-a_n)(a_m-a_p)(a_n-a_p)(a_k-a_q)(a_k-a_r)(a_k-a_s)(a_k-a_t)]}{(a_m-a_q)(a_n-a_q)(a_p-a_q)(a_k-a_l)}.$$

Multiplying then 23 by the factor

$$(a_m-a_q)(a_n-a_q)(a_p-a_q)(a_k-a_l),$$

and 24 by the factor $(a_l-a_m)(a_l-a_n)(a_l-a_p)(a_k-a_p)$,
and subtracting one result from the other we eliminate P_0 and have

$$\begin{aligned}&P_{kl}^2|a_ma_na_pa_q||a_ka_l|\\&-P_{km}^2\{|a_na_pa_l||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_na_pa_q||a_la_m||a_la_n||a_la_p||a_ka_p|\}\\&+P_{kn}^2\{|a_pa_la_m||a_mq_q||a_na_q||a_pa_q||a_ka_l|+|a_pa_qa_m||a_la_m||a_la_n||a_la_p||a_ka_p|\}\\&-P_{kp}^2\{|a_la_ma_n||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_qa_ma_n||a_la_m||a_la_n||a_la_p||a_ka_p|\}\\&+P_{kq}^2\{|a_ma_na_p||a_la_m||a_la_n||a_la_p||a_ka_p|\}=\\&P_k^2\{a_la_ma_na_p||a_ma_q||a_na_q||a_pa_q||a_ka_l|-|a_ma_na_pa_q||a_la_m||a_la_n||a_la_p||a_ka_p|\}\end{aligned}$$

This reduces to

$$\begin{aligned}&P_{kl}^2|a_ma_na_pa_q||a_ka_l|-P_{km}^2|a_na_pa_qa_l||a_ka_m|+P_{kn}^2|a_pa_qa_la_m||a_ka_n|\\ 25.\quad &-P_{kp}^2|a_qa_la_ma_n||a_ka_p|+P_{kq}^2|a_la_ma_na_p||a_ka_q|=\\&P_k^2|a_la_ma_na_pa_q|.\end{aligned}$$

If we here make again the substitution

$$\begin{vmatrix} l & m & n & p & q & r & s & t\\ m & n & p & q & r & s & t & l\end{vmatrix}$$

we get a new relation connecting the squares of P_k, P_{km}, P_{kn}, P_{kp}, P_{kq}, P_{kr}. The coefficient of P_k^2 will be $|a_ma_na_pa_qa_r|$:
multiplying this equation then by the factor

$$|a_la_m||a_la_n||a_la_p||a_la_q|,$$

and multiplying 25 by $|a_ma_r||a_na_r||a_pa_r||a_qa_r|$,
and subtracting one result from the other, we eliminate P_k and have a linear relation between the squares of

$$P_{kl},\ P_{km},\ P_{kn},\ P_{kp},\ P_{kq},\ P_{kr},\ \text{viz.}$$

$$\begin{aligned}&P_{kl}^2|a_ma_na_pa_qa_r||a_ka_l|-P_{km}^2|a_na_pa_qa_ra_l||a_ka_m|\\ 26.\quad &+P_{kn}^2|a_pa_qa_ra_la_m||a_ka_n|-P_{kp}|a_qa_ra_la_ma_n||a_ka_p|\\&+P_{kq}^2|a_ra_la_ma_na_p||a_ka_q|-P_{kr}|a_la_ma_na_pa_q||a_ka_r|.\end{aligned}$$

It is obvious that we might have eliminated P_k^2 between 23 and 24, and so have found a linear relation connecting the squares of

$$P_0,\ P_{kl},\ P_{km},\ P_{kn},\ P_{kp},\ P_{kq},$$

and by making the above substitution and eliminating P_0^2 between the two equations thus formed we would again arrive at 26. It is then clear that the functions with the indices

$$0,\ k,\ kl,\ km,\ kn,\ kp,\ kq,\ kr,\ ks,\ kt,$$

form a group of ten, such that any five being selected the squares of any of the remaining five can be expressed as a linear function of the squares of the chosen five. There are of course in all nine such groups, and these may be tabulated as follows:

0	*k*	*kl*	*km*	*kn*	*kp*	*kq*	*kr*	*ks*	*kt*
0	*l*	*lk*	*lm*	*ln*	*lp*	*lq*	*lr*	*ls*	*lt*
0	*m*	*mk*	*ml*	*mn*	*mp*	*mq*	*mr*	*ms*	*mt*
0	*n*	*nk*	*nl*	*nm*	*np*	*nq*	*nr*	*ns*	*nt*
0	*p*	*pk*	*pl*	*pm*	*pn*	*pq*	*pr*	*ps*	*pt*
0	*q*	*qk*	*ql*	*qm*	*qn*	*qp*	*qr*	*qs*	*qt*
0	*r*	*rk*	*rl*	*rm*	*rn*	*rp*	*rq*	*rs*	*rt*
0	*s*	*sk*	*sl*	*sm*	*sn*	*sp*	*sq*	*sr*	*st*
0	*t*	*tk*	*tl*	*tm*	*tn*	*tp*	*tq*	*tr*	*ts*

and the groups will be called the k-group, the l-group, etc.

We will now take up the case of three indices, which, as will be seen, divides into two sub-cases, according to the choice of the index. The two sub-cases give rise to two tables, the first containing 36 groups and the second containing 84 groups. As the method of working out these groups by Brunel's method has already been sufficiently indicated, I shall, in what follows, leave out as much as possible the purely algebraical processes of reduction, as they now become very long and wholly uninteresting. Squaring the function P_{klm} we have

$$\begin{aligned}27.\quad P_{klm}^2=\frac{1}{|x_1x_2x_3x_4|^2}\Big\{&|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|}+\ldots\\&-2|x_2x_3x_4|\sqrt{R(x_1)R(x_2)}\,|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_mx_3||a_mx_4|+\ldots-\ldots+\ldots\Big\}\end{aligned}$$

The radicals $\sqrt{R(x_1)R(x_2)}$, etc., may be eliminated between any six equations of the form 27, or between five equations of this form, having each a common index, say k, or between four equations having each two indices, say k and l, common. Choose multipliers A, B, C and D, such that the coefficient of $\sqrt{R(x_1)R(x_2)}$ (and in consequence the coefficients of all the other radicals) shall be zero in the sum $AP_{klm}^2+BP_{kln}^2+CP_{klp}^2+DP_{kl}^2$.
This coefficient is easily seen to be

$$\begin{aligned}&A|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_mx_3||a_mx_4|+B|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_nx_3||a_nx_4|\\&+C|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_px_3||a_px_4|+D|a_kx_3||a_kx_4||a_lx_3||a_lx_4|\end{aligned}$$

Striking out the common factor

$$|a_k x_3||a_k x_4||a_l x_3||a_l x_4|$$

the condition to be satisfied is

28. $$A|a_m x_3||a_m x_4| + B|a_n x_3||a_n x_4| + C|a_p x_3||a_p x_4| + D = 0;$$

this is equivalent to

$$A + B + C = 0,$$
$$a_m A + a_n B + a_p C = 0,$$
$$a_m^2 A + a_n^2 B + a_p^2 C + D = 0.$$

These are easily seen to be satisfied by the values

29. $$A = |a_n a_p|,\ B = |a_p a_m|,\ C = |a_m a_n|,\ D = -|a_m a_n a_p|.$$

Introducing then these values of A, B, C and D, we have

$$|a_n a_p|P^2_{klm} + |a_p a_m|P^2_{kln} + |a_m a_n|P^2_{klp} - |a_m a_n a_p|P^2_{kl} =$$

30. $$\frac{1}{|x_1 x_2 x_3 x_4|^2}\Big\{|a_n a_p|\Big[|x_2 x_3 x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4|}{|a_k x_1||a_l x_1||a_m x_1|} + \ldots\Big]$$
$$+|a_p a_m|\Big[|x_2 x_3 x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_n x_2||a_n x_3||a_n x_4|}{|a_k x_1||a_l x_1||a_n x_1|} + \ldots\Big]$$
$$+|a_m a_n|\Big[|x_2 x_3 x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_p x_2||a_p x_3||a_p x_4|}{|a_k x_1||a_l x_1||a_p x_1|} + \ldots\Big]$$
$$-|a_m a_n a_p|\Big[|x_2 x_3 x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|} + \ldots\Big]\Big\}$$

This is to be reduced just as in the case of two indices, viz. expand 30, so that the first line becomes

$$\{|a_n a_p||x_2 x_3 x_4||a_n x_1||a_p x_1|\ldots|a_l x_1||a_k x_2||a_k x_3||a_k x_4||a_l x_2|\ldots|a_l x_4|\}$$

31. $$\times\{|x_2 x_3 x_4||a_m x_2||a_m x_3||a_m x_4| - |x_3 x_4 x_1||a_m x_3||a_m x_4||a_m x_1| + \ldots$$
$$- |x_1 x_2 x_3||a_m x_1||a_m x_2||a_m x_3|\} + \ldots$$

There are three more terms similar to this to be obtained by simply advancing certain of the subscripts. The remaining three lines in 30 are to be expanded in a similar manner, and then the terms which have been introduced will disappear by aid of equations 28 and 29. The right-hand side of 30 is now easily reduced by aid of the following identities:

$$|x_2 x_3 x_4| - |x_3 x_4 x_1| + |x_4 x_1 x_2| - |x_1 x_2 x_3| = 0,$$

32. $$(x_2 + x_3 + x_4)|x_2 x_3 x_4| - (x_3 + x_4 + x_1)|x_3 x_4 x_1|$$
$$+ (x_4 + x_1 + x_2)|x_4 x_1 x_2| - (x_1 + x_2 + x_3)|x_1 x_2 x_3| = 0,$$

$$(x_2 x_3 + x_3 x_4 + x_4 x_2)|x_2 x_3 x_4| - (x_3 x_4 + x_4 x_1 + x_1 x_3)|x_3 x_4 x_1|$$
$$+ (x_4 x_1 + x_1 x_2 + x_2 x_4)|x_4 x_1 x_2| - (x_1 x_2 + x_2 x_3 + x_3 x_1)|x_1 x_2 x_3| = 0,$$

$$x_2 x_3 x_4|x_2 x_3 x_4| - x_3 x_4 x_1|x_3 x_4 x_1| + x_4 x_1 x_2|x_4 x_1 x_2| - x_1 x_2 x_3|x_1 x_2 x_3| = -|x_1 x_2 x_3 x_4|.$$

A similar group of identities may easily be written down for the general case. Using these last identities and taking the four functions

$$P^2_{klm},\ P^2_{kln},\ P^2_{klp},\ P^2_{klq},$$

we can, by multiplying the first by A, the second by B, etc., and taking the sum, eliminate the radicals $\sqrt{R(x_1)R(x_2)}$, $\sqrt{R(x_1)R(x_3)}$, etc. Forming this sum it is only necessary to show that

39. $$A|a_mx_3||a_mx_4|+B|a_nx_3||a_nx_4|+C|a_px_3||a_px_4|+D|a_qx_3||a_qx_4|=0;$$

and this equation of condition is at once seen to be satisfied by the values

39′. $$A=|a_na_pa_q|,\ B=-|a_pa_qa_m|,\ C=|a_qa_ma_n|,\ D=-|a_ma_na_p|.$$

We have now

$$|a_na_pa_q|P^2_{klm}-|a_pa_qa_m|P^2_{kln}+|a_qa_ma_n|P^2_{klp}-|a_ma_na_p|P^2_{klq}=$$

40. $$\frac{1}{|x_1x_2x_3x_4|^2}\Big\{|a_na_pa_q|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|}+\dots\Big]$$
$$-|a_pa_qa_m|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_nx_2||a_nx_3||a_nx_4|}{|a_kx_1||a_lx_1||a_nx_1|}+\dots\Big]$$
$$+|a_qa_ma_n|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_px_2||a_px_3||a_px_4|}{|a_kx_1||a_lx_1||a_px_1|}+\dots\Big]$$
$$-|a_ma_na_p|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_qx_2||a_qx_3||a_qx_4|}{|a_kx_1||a_lx_1||a_qx_1|}+\dots\Big]\Big\}.$$

This may be briefly written in the form

$$|a_na_pa_q|P^2_{klm}-|a_pa_qa_m|P^2_{kln}+|a_qa_ma_n|P^2_{klp}-|a_ma_na_p|P^2_{klq}=$$

41. $$\frac{1}{|x_1x_2x_3x_4|^2}\Big\{|a_na_pa_q|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,a_l,a_m][x_2,x_3,x_4]}{[a_k,a_l,a_m][x_1]}+\dots\Big]$$
$$-|a_pa_qa_m|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,a_l,a_n][x_2,x_3,x_4]}{[a_k,a_l,a_n][x_1]}+\dots\Big]$$
$$+|a_qa_ma_n|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,a_l,a_p][x_2,x_3,x_4]}{[a_k,a_l,a_p][x_1]}+\dots\Big]$$
$$-|a_ma_na_p|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,a_l,a_q][x_2,x_3,x_4]}{[a_k,a_l,a_q][x_1]}+\dots\Big]\Big\}$$

Expanding this just as in the case of two indices and the case of equation 30, we have for the first line on the right-hand side of the equation

$$\{|a_na_pa_q||x_2x_3x_4|[a_n,a_p,a_q,a_r,a_s,a_t][x_1][a_k,a_l][x_2,x_3,x_4]\}$$
$$\times\{|x_2x_3x_4|[a_m][x_2,x_3,x_4]-|x_3x_4x_1|[a_m][x_3,x_4,x_1]$$
$$+|x_4x_1x_2|[a_m][x_4,x_1,x_2]-|x_1x_2x_3|[a_m][x_1,x_2,x_3]\}$$

$+$ three similar terms.

The remaining three lines of 41 are to be expanded in the same manner, and then it will at once be seen that the extra terms which have been introduced

will vanish on account of the relations 39 and 39′. Consider now the terms containing the factor $R(x_1)$: they have obviously the common factor

$$|x_2x_3x_4|[a_r,a_s,a_t][x_1].[a_k,a_l][x_2,x_3,x_4]$$

and the remaining factor is

$$|a_na_pa_q|[a_n,a_p,a_q][x_1]-|a_pa_qa_m|[a_p,a_q,a_m][x_1]$$
$$+|a_qa_ma_n|[a_q,a_m,a_n][x_1]-|a_ma_na_p|[a_m,a_n,a_p][x_1]=,\text{ say }K.$$

Expanding K and using the identities 32 and 33, we have

42. $$K=-|a_ma_na_pa_q|.$$

The first line on the right-hand side of equations 40 or 41 contains four terms, the first of which contains the factor already mentioned, viz.

43. $$|x_2x_3x_4|[a_m][x_2,x_3,x_4]-|x_3x_4x_1|[a_m][x_3,x_4,x_1]+|x_4x_1x_2|[a_m][x_4,x_1,x_2]$$
$$-|x_1x_2x_3|[a_m][x_1,x_2,x_3];$$

the factor in each of the remaining terms is derived from this by changing m into n, p, q respectively. The same is true for the remaining three lines on the right-hand side of 40 or 41. This factor is independent of a_m, and the others not written down are equally independent of n, p, or q; for writing 43 out in full it is

$$|x_2x_3x_4||a_mx_2||a_mx_3||a_mx_4|-|x_3x_4x_1||a_mx_3||a_mx_4||a_mx_1|$$
$$+|x_4x_1x_2||a_mx_4||a_mx_1||a_mx_2|-|x_1x_2x_3||a_mx_1||a_mx_2||a_mx_3|$$

and this is equal to

$$a^3_m\Sigma|x_2x_3x_4|+a^2_m\Sigma(x_2+x_3+x_4)|x_2x_3x_4|$$
$$+a_m\Sigma(x_2x_3+x_3x_4+x_4x_2)|x_2x_3x_4|+\Sigma x_2x_3x_4|x_2x_3x_4|.$$

The first three terms of this vanish by virtue of the identities 32, and the fourth term by 33 becomes $=-|x_1x_2x_3x_4|$.

The right-hand side of 40 and 41 thus contai the factor

$$-|a_ma_na_pa_q|.-|x_1x_2x_3x_4|=|a_ma_na_pa_q||x_1x_2x_3x_4|.$$

The right-hand member of 41 takes now the form

$$\frac{1}{|x_1x_2x_3x_4|}|a_ma_na_pa_q|\Sigma|x_2x_3x_4|[a_r,a_s,a_t][x_1][a_k,a_l][x_2,x_3,x_4].$$

The Σ of course refers only to the cyclic per itations of the suffixes 1, 2, 3, 4. We have now to determine the value of the quantity under the summation sign, viz.

$$\Sigma|x_2x_3x_4||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|$$

in order to find the relation connecting the squares of P_{klm}, P_{kln}, P_{klp}, P_{klq}. For greater generality and completeness, however, it is better to go back to equation 30 and reduce it, *i. e.* find the linear relation connecting the squares of P_{klm}, P_{kln}, P_{klp}, P_{kl}. It will then be seen that by making the substitution

$$\begin{vmatrix} k & l & m & n & p & q & r & s & t \\ l & m & n & p & q & r & s & t & k \end{vmatrix}$$

and eliminating one quantity, we arrive at what we might obtain directly by completing the reduction of equation 40.

The second factor in 31 is easily seen to be

44. $$= -|x_1x_2x_3x_4|,$$

and the same is true for the corresponding factors in the remaining three lines of 30. Now adding together the first factors of the four lines in 30, viz. those similar to the first factor in 31, we have

$$\begin{aligned}&|a_na_p||x_2x_3x_4|[a_n, a_p \dots a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4]\\ &+|a_pa_m||x_2x_3x_4|[a_p, a_q \dots a_t, a_m][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4]\\ &+|a_ma_n||x_2x_3x_4|[a_m, a_p \dots a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4].\end{aligned}$$

The fourth line need not be written down, as its second factor is zero.

Adding these terms we have

$$\begin{aligned}&\Sigma\{|x_2x_3x_4|[a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4]\\ &\times|a_na_p||a_nx_1||a_px_1|+|a_pa_m||a_px_1||a_mx_1|+|a_ma_n||a_mx_1||a_nx_1|\}.\end{aligned}$$

Now $$|a_na_p|+|a_pa_m|+|a_ma_n|=0,$$

$$(a_n+a_p)|a_na_p|+(a_p+a_m)|a_pa_n|+(a_m+a_n)|a_ma_n|=0,$$

and $$a_na_p|a_na_p|+a_pa_m|a_pa_m|+a_ma_n|a_ma_n|=|a_ma_na_p|,$$

so that the above reduces to

45. $$|a_ma_na_p|\Sigma|x_2x_3x_4|[a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4],$$

the summation referring to the subscripts 1, 2, 3, 4. Equation 30, or equation 31, becomes now, by taking into acccount 44 and 45,

46. $$\begin{aligned}&|a_na_p|P^2_{klm}+|a_pa_m|P^2_{kln}+|a_ma_n|P^2_{klp}-|a_ma_na_p|P^2_{kl}=\\ &\frac{1}{|x_1x_2x_3x_4|}|a_ma_na_p|\Sigma|x_2x_3x_4|[a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4].\end{aligned}$$

For brevity write as before

$$\Sigma a_q=\lambda,\ \Sigma a_qa_r=\mu,\ \Sigma a_qa_ra_s=\nu,\ a_qa_ra_sa_t=\pi,$$

the summations extending over the subscripts q, r, s, t. We have now

47. $$\begin{aligned}&\Sigma|x_2x_3x_4|[a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4]\\ &=\Sigma|x_2x_3x_4|(\pi-x_1\nu+x_1^2\mu-x_1^3\lambda+x_1^4)\{(a_k^3-a_k^2(x_2+x_3+x_4)+a_k(x_2x_3+x_3x_4+x_4x_2)-x_2x_3x_4)\\ &\qquad\times(a_l^3-a_l^2(x_2+x_3+x_4)+a_l(x_2x_3+x_3x_4+x_4x_2)-x_2x_3x_4)\}\\ &=\Sigma|x_2x_3x_4|(\pi-x_1\nu+x_1^2\mu-x_1^3\lambda+x_1^4)\{a_k^3a_l^3-a_k^2a_l^2(a_k+a_l)(x_2+x_3+x_4)\\ &\quad+a_k^2a_l^2(x_2+x_3+x_4)^2+a_ka_l(a_k^2+a_l^2)(x_2x_3+x_3x_4+x_4x_2)\\ &\quad-a_ka_l(a_k+a_l)(x_2+x_3+x_4)(x_2x_3+x_3x_4+x_4x_2)+a_ka_l(x_2x_3+x_3x_4+x_4x_2)^2\\ &\quad+(a_k^2+a_l^2)(x_2+x_3+x_4)x_2x_3x_4-(a_k+a_l)(x_2x_3+x_3x_4+x_4x_2)x_2x_3x_4\\ &\quad-(a_k^3+a_l^3)x_2x_3x_4+x_2^2x_3^2x_4^2\}.\end{aligned}$$

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If instead of eliminating P^2_{kl} between 48 and 49 we had eliminated P^2_l, we would arrive at a linear relation connecting

$$P^2_{klm},\ P^2_{kln},\ P^2_{klp},\ P^2_{klq},\ P^2_{kt},\ P^2_k,$$

say

51. $$AP^2_{klm}+BP^2_{kln}+CP^2_{klp}+DP^2_{klq}+EP^2_{kl}+FP^2_k=0.$$

Now effect upon 50 the substitution

$$\begin{vmatrix} m & n & p & q & r & s & t \\ n & p & q & r & s & t & m \end{vmatrix} = \text{say } \Omega,$$

then we will obviously obtain a relation of the form,

52. $$A'P^2_{kln}+B'P^2_{klp}+C'P^2_{klq}+D'P_{klr}+E'P^2_{kl}+F'P^2_k=0.$$

Eliminating P^2_{kl} between 51 and 52 and we have a linear relation connecting five of the P-functions possessing triple indices, and one possessing a single index, viz., a relation of the form

53. $$A''P^2_{klm}+B''P^2_{kln}+C''P^2_{klp}+D''P^2_{klq}+E''P^2_{klr}+F''P^2_k=0.$$

Or, if P^2_k had been eliminated, we would have a relation connecting five P-functions with triple indices, and one with a double index, viz.

54. $$A'''P^2_{klm}+B'''P^2_{kln}+C'''P^2_{klp}+D'''P^2_{klq}+E'''P^2_{klr}+F'''P^2_{kl}=0.$$

Effecting the substitution Ω upon 54 and there results an equation of the form,

55. $$A^{\mathrm{iv}}P^2_{kln}+B^{\mathrm{iv}}P^2_{klp}+C^{\mathrm{iv}}P^2_{klq}+D^{\mathrm{iv}}P^2_{klr}+E^{\mathrm{iv}}P^2_{kls}+F^{\mathrm{iv}}P^2_{kl}=0.$$

Now finally eliminating P^2_{kl} between 54 and 55 and we arrive at a linear relation connecting the squares of six P-functions with triple indices, viz.

56. $$A^{\mathrm{v}}P^2_{klm}+B^{\mathrm{v}}P^2_{kln}+C^{\mathrm{v}}P^2_{klp}+D^{\mathrm{v}}P^2_{klq}+E^{\mathrm{v}}P^2_{klr}+F^{\mathrm{v}}P^2_{kls}=0.$$

Of course, the substitution Ω performed upon 56 would give

57. $$A^{\mathrm{vi}}P^2_{kln}+B^{\mathrm{vi}}P^2_{klp}+C^{\mathrm{vi}}P^2_{klq}+D^{\mathrm{vi}}I\ \ +E^{\mathrm{vi}}P^2_{kls}+F^{\mathrm{vi}}P^2_{klt}=0.$$

We arrive thus at the conclusion that the ten functions

$$P_k,\ P_l,\ P_{kl},\ P_{klm},\ P_{kln},\ P_{klp},\ P_{klq},\ P_{klr},\ P_{kls},\ P_{klt}$$

form a group such that selecting any five of them the square of any one of the remaining five is linearly expressible in terms of the squares of the chosen five. There are in all 36 such groups, and they are given in the following table:

k	*l*	*kl*	*klm*	*kln*	*klp*	*klq*	*klr*	*kls*	*klt*
k	*m*	*km*	*kml*	*kmn*	*kmp*	*kmq*	*kmr*	*kms*	*kmt*
k	*n*	*kn*	*knl*	*knm*	*knp*	*knq*	*knr*	*kns*	*knt*
k	*p*	*kp*	*kpl*	*kpm*	*kpn*	*kpq*	*kpr*	*kps*	*kpt*
k	*q*	*kq*	*kql*	*kqm*	*kqn*	*kqp*	*kqr*	*kqs*	*kqt*
k	*r*	*kr*	*krl*	*krm*	*krn*	*krp*	*krq*	*krs*	*krt*
k	*s*	*ks*	*ksl*	*ksm*	*ksn*	*ksp*	*ksq*	*ksr*	*kst*

k	*t*	*kt*	*ktl*	*ktm*	*ktn*	*ktp*	*ktq*	*ktr*	*kts*
l	*m*	*lm*	*lmk*	*lmn*	*lmp*	*lmq*	*lmr*	*lms*	*lmt*
l	*n*	*ln*	*lnk*	*lnm*	*lnp*	*lnq*	*lnr*	*lns*	*lnt*
l	*p*	*lp*	*lpk*	*lpm*	*lpn*	*lpq*	*lpr*	*lps*	*lpt*
l	*q*	*lq*	*lqk*	*lqm*	*lqn*	*lqp*	*lqr*	*lqs*	*lqt*
l	*r*	*lr*	*lrk*	*lrm*	*lrn*	*lrp*	*lrq*	*lrs*	*lrt*
l	*s*	*ls*	*lsk*	*lsm*	*lsn*	*lsp*	*lsq*	*lsr*	*lst*
l	*t*	*lt*	*ltk*	*ltm*	*ltn*	*ltp*	*ltq*	*ltr*	*lts*
m	*n*	*mn*	*mnk*	*mnl*	*mnp*	*mnq*	*mnr*	*mns*	*mnt*
m	*p*	*mp*	*mpk*	*mpl*	*mpn*	*mpq*	*mpr*	*mps*	*mpt*
m	*q*	*mq*	*mqk*	*mql*	*mqn*	*mqp*	*mqr*	*mqs*	*mqt*
m	*r*	*mr*	*mrk*	*mrl*	*mrn*	*mrp*	*mrq*	*mrs*	*mrt*
m	*s*	*ms*	*msk*	*msl*	*msn*	*msp*	*msq*	*msr*	*mst*
m	*t*	*mt*	*mtk*	*mtl*	*mtn*	*mtp*	*mtq*	*mtr*	*mts*
n	*p*	*np*	*npk*	*npl*	*npm*	*npq*	*npr*	*nps*	*npt*
n	*q*	*nq*	*nqk*	*nql*	*nqm*	*nqp*	*nqr*	*nqs*	*nqt*
n	*r*	*nr*	*nrk*	*nrl*	*nrm*	*nrp*	*nrq*	*nrs*	*nrt*
n	*s*	*ns*	*nsk*	*nsl*	*nsm*	*nsp*	*nsq*	*nsr*	*nst*
n	*t*	*nt*	*ntk*	*ntl*	*ntm*	*ntp*	*ntq*	*ntr*	*nts*
p	*q*	*pq*	*pqk*	*pql*	*pqm*	*pqn*	*pqr*	*pqs*	*pqt*
p	*r*	*pr*	*prk*	*prl*	*prm*	*prn*	*prq*	*prs*	*prt*
p	*s*	*ps*	*psk*	*psl*	*psm*	*psn*	*psq*	*psr*	*pst*
p	*t*	*pt*	*ptk*	*ptl*	*ptm*	*ptn*	*ptq*	*ptr*	*pts*
q	*r*	*qr*	*qrk*	*qrl*	*qrm*	*qrn*	*qrp*	*qrs*	*qrt*
q	*s*	*qs*	*qsk*	*qsl*	*qsm*	*qsn*	*qsp*	*qsr*	*qst*
q	*t*	*qt*	*qtk*	*qtl*	*qtm*	*qtn*	*qtp*	*qtr*	*qts*
r	*s*	*rs*	*rsk*	*rsl*	*rsm*	*rsn*	*rsp*	*rsq*	*rst*
r	*t*	*rt*	*rtk*	*rtl*	*rtm*	*rtn*	*rtp*	*rtq*	*rts*
s	*t*	*st*	*stk*	*stl*	*stm*	*stn*	*stp*	*stq*	*str*

Consider now the functions with four indices and find values of A, B, C, D, such that the radicals $\sqrt{R(x_1)R(x_2)}$, etc. shall vanish in the sum

$$AP^2_{klmn} - BP^2_{klmp} + CP^2_{klmq} - DP^2_{klmr}.$$

The coefficient of $\sqrt{R(x_1)R(x_2)}$ in this sum is, leaving out the common factor $|x_2x_3x_4||x_3x_4x_1|$,

$$A[a_k, a_l, a_m, a_n][x_3][a_k, a_l, a_m, a_n][x_4] - B[a_k, a_l, a_m, a_p][x_3][a_k, a_l, a_m, a_p][x_4]$$
$$+ C[a_k, a_l, a_m, a_q][x_3][a_k, a_l, a_m, a_q][x_4] - D[a_k, a_l, a_m, a_r][x_3][a_k, a_l, a_m, a_r][x_4] = 0.$$

Taking out the common factor

$$[a_k, a_l, a_m][x_3][a_k, a_l, a_m][x_4],$$

and this becomes

$$A|a_n x_3||a_n x_4| - B|a_p x_3||a_p x_4| + C|a_q x_3||a_q x_4| - D|a_r x_3||a_r x_4| = 0,$$

giving

$$A = |a_p a_q a_r|, \quad B = |a_q a_r a_n|,$$
$$C = |a_r a_n a_p|, \quad D = |a_n a_p a_q|.$$

Introducing here the values

$$R(x_1) = [a_k, a_l, a_m, a_n, a_p, a_q, a_r, a_s, a_t][x_1], \text{ etc.}$$

we ought to be able to show that the squares of any six of the functions whose indices are *kl lm mk klm klmn klmp klmq klmr klms klmt* are connected by a linear relation. We would then have a table of 84 groups similar to the above, and such that the squares of any six functions in a given group are connected by a linear relation. In the case of quadruple indices there would also be a second table containing 126 groups, of which

klm lmn mnk nkl klmn pqrs qrst rstp stpq tpqr

is the first, and the squares of any six of these functions should also be connected by a linear relation.

We would thus have in all 256 groups giving linear relations between the squares of six P-functions. There would be 840 relations of this kind, but not all, of course, asyzygetic.

These 256 groups of ten functions each might be called the 256 decads, and they would obviously correspond to the 16 Kummer hexads in the case of the double theta-functions, viz. between the squares of any four theta-functions of a Kummer hexad there exists a linear relation, and between the squares of any six of the P-functions belonging to a given decad there exists a linear relation, the same kind of relation will obviously exist between the squares of the six corresponding quadruple theta-functions. We have thus hexads of double theta-functions of which the squares of any four are connected by a linear relation; octads of triple theta-functions of which the squares of any five are connected by a linear relation; decads of quadruple theta-functions of which the squares of any six are connected by a linear relation, and in general $2(p+1)$-ads of p-tuple theta-functions of which the squares of any $p+2$ are connected by a linear relation. It seems highly probable that this generalization is true, but I have not as yet been able to prove it.

In order to show the linear relations between the squares of the P-functions with quadruple indices, we will begin with the 84 groups, of which

$$kl\ lm\ mk\ klm\ klmn\ klmp\ klmq\ klmr\ klms\ klmt$$

is the first. Take the functions

$$P_{klm},\ P_{klmn},\ P_{klmp},\ P_{klmq},$$

and find the values of A, B, C, D, so that the radicals $\sqrt{R(x_1)R(x_2)}$, etc., shall vanish in the sum $AP^2_{klm}+BP^2_{klmn}+CP^2_{klmp}+DP^2_{klmq}$.

Dropping out a common factor

$$[a_k][x_3,\ x_4][a_l][x_3,\ x_4][a_m][x_3,\ x_4]$$

the necessary condition is easily seen to be

$$A+B[a_n][x_3,\ x_4]+C[a_p][x_3,\ x_4]+D[a_q][x_3,\ x_4]=0\,;$$

this is equivalent to

$$B+C+D=0,$$

$$a_nB+a_pC+a_qD=0,$$

$$A+a_n^2B+a_p^2C+a_q^2D=0,$$

and these are easily seen to be satisfied by the values

$$A=-|a_na_pa_q|,\quad B=|a_pa_q|,\quad C=|a_qa_n|,\quad D=|a_na_p|.$$

Introducing these values in the above sum we have

$$-|a_na_pa_q|P^2_{klm}+|a_pa_q|P^2_{klmn}+|a_qa_n|P^2_{klmp}+|a_na_p|P^2_{klmq}$$

$$=\frac{1}{|x_1x_2x_3x_4|^2}$$

$$\{-|a_na_pa_q|[|x_2x_3x_4|^2[a_n,\ a_p,\ a_q,\ a_r,\ a_s,\ a_t][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]+\ldots]$$

$$+|a_pa_q|[|x_2x_3x_4|^2[a_p,a_q,a_r,a_s,a_t][x_1][a_k][x_2,x_3,x_4][a_l][x_2,x_3,x_4][a_m][x_2,x_3,x_4][a_n][x_2,x_3,x_4]+\ldots]$$

$$+|a_qa_n|[|x_2x_3x_4|^2[a_q,a_r,a_s,a_t,a_n][x_1]\text{ `` `` `` `` `` `` `` `` }]+\ldots]$$

$$+|a_na_p|[|x_2x_3x_4|^2[a_r,a_s,a_t,a_n,a_p][x_1]\text{ `` `` `` `` `` `` `` `` }]+\ldots]\}$$

The right-hand side of this equation may be written in the form

$$\frac{1}{|x_1x_2x_3x_4|^2}[|a_pa_q||x_2x_3x_4|[a_r,\ a_s,\ a_t,\ a_p,\ a_q][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]$$

$$\times\{|x_2x_3x_4|[a_n][x_2,\ x_3,\ x_4]-|x_3x_4x_1|[a_n][x_3,\ x_4,\ x_1]+|x_4x_1x_2|[a_n][x_2,\ x_3,\ x_4]$$

$$-|x_1x_2x_3|[a_n][x_1,\ x_2,\ x_3]\}+\ldots$$

+ (two similar terms containing $[a_q,\ a_n][x_1]$, $[a_n,\ a_p][x_1]$ respectively)

$$-|a_na_pa_q|[a_r,\ a_s,\ a_t,\ a_n,\ a_p,\ a_q][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]$$

$$\times\{|x_2x_3x_4|-|x_3x_4x_1|+|x_4x_1x_2|-|x_1x_2x_3|\}+\ldots]$$

The omitted terms are easily supplied by symmetry. Now

$$\Sigma|x_2x_3x_4|=0,\quad \Sigma|x_2x_3x_4|[a_n][x_2,\ x_3,\ x_4]=|x_1x_2x_3x_4|$$

The fourth line of the last equation vanishes and the first terms of the first three

$$|x_2x_3x_4|[a_r,\ a_s,\ a_t][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]$$

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