[These things are worth saying; albeit the first question, upon which our correspondent so lightly expresses himself, involves points of high debate among those who of late years have studied the problem of genius. We shall return to some of these in a notice of Lombroso's 'Man of Genius.' Meantime, we remark that Kepler, with all his advantages, did but one great work, and that that was by no means the sort of thing a college tutor "would find time enough" to toss off.

As to the teaching of the two important branches of mathematics to which Mr. Stille refers, we derive the following information, relating chiefly to the year 1888-9, from Prof. Cajori's document on the 'Teaching of Mathematics in the United States.' The Theory of Functions was the subject, at the Johns Hopkins University, of several courses; at Cornell of a two-years' elective course, with sessions thrice a week one year, twice a week another; at Harvard of an advanced course; at Princeton of a University course; at Madison of a "special advanced elective" (possibly not taken). Projective (modern synthetic) Geometry was the subject of a course at the Johns Hopkins; at Cornell was *required* for some students, elective for others; at Ann Arbor was studied (and *really* studied, as we happen to know) from Reye's admirable treatise; and at the Universities of Texas (where Cremona's charming book was used), Virginia, and South Carolina formed the subject of post-graduate lectures or examinations. This, though a poor showing, yet makes a beginning.—ED. NATION.]

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THE NON-EUCLIDEAN GEOMETRY

Geometrical Researches on the Theory of Parallels.

By Nicholaus Lobatchewsky. Translated from the original by George Bruce Halsted, A.M., Ph.D., ex-fellow of Princeton College and Johns Hopkins University, Professor of Mathematics in the University of Texas. Austin. 1891.

CSP. identification: MS 1365; Haskell, *Index to* The Nation. See also: Burks, *Bibliography: List of Articles.*

Nikolai Ivanovich Lobachevski (1793-1856) was a Russian mathematician. He demonstrated his genius in mathematics at the University of Kazan, where ultimately he became president. He published his geometry in 1829, after holding it back for several years. Philosophically, the development of non-Euclidean geometry shattered the notion of self-evident truth in its most secure stronghold, mathematics.

George Bruce Halsted (1853-1922) was a mathematician of considerable fame. He took his Ph.D. from Johns Hopkins in 1879 (where he was the first student of J. J. Sylvester), and in 1884 moved to the University of Texas, where he assumed the post of professor of mathematics. While at the University of Texas, he published 25 books on mathematics, and authored numerous articles for journals such as *The Monist, The Educational Review,* and *The Popular Science Monthly.*

Lobachevski's little book, 'Geometrische Untersuchungen,' marks an epoch in the history of thought, that of the overthrow of the axioms of geometry. The philosophical consequences of this are undoubtedly momentous, and there are thinkers who hold that it must lead to a new conception of nature, less mechani-

cal than that which has guided the steps of science since Newton's discovery. The book has been published many years—in fact, the essence of it was set forth before 1830; so long does it take a pure idea to make its way, unbacked by any interest more aggressive than the love of truth. In this case, the idea is lucid, easy, and convincing. Nobody with enough mathematical capacity to be able to understand the first book of geometry need fear the least difficulty in mastering Lobachevski's tract; and really it is high time that every thinking man and woman should know what is in it.

In the pre-Lobachevskian days, elementary geometry was universally regarded as the very exemplar of conclusive reasoning carried to great lengths. It had been the ideal of speculative thinkers in all ages. Metaphysics, indeed, as an historical fact, has been nothing but an attempt to copy, in thinking about substances, the geometer's reasoning about shapes. This is shown by the declarations of Plato and others, by the spatial origin of many metaphysical conceptions and of the terms appropriated to them, such as *abstract, form, analogy*, etc., and by the love of donning the outer clothing of geometry, even when no fit for philosophy. For instance, one of the remarkable features of geometry is the small number of premises from which galaxies of theorems result; and accordingly it has been the effort of almost all metaphysicians to reduce their first principles to the fewest possible, even if they had to crowd disparate thoughts into one formula. It did not seem to occur to them that since a list of first principles is a work of analysis, it would not be a small number of elementary propositions so much as a large number that would bespeak its thoroughness. Admiration for the elements of geometry was not, however, confined to metaphysicians. Euclid's treatise was acknowledged by all kinds of minds to be all but absolutely perfect in its reasoning, and the very type of what science should aim at as to form and matter.

In the empyrean of geometry there was but one little speck—the theory of parallels. Euclid had had a difficulty in proving the sum of the angles of a triangle to be not less than two right angles. His treatment of the subject betrays a very profound study of it; for instead of slipping over the difficulty unaware, as forty-nine out of fifty mathematicians would have done, instead of even bringing the necessary assumption to a persuasive shape, he takes as his fifth postulate (or 11th axiom, in incorrect editions) a proposition that begs the question in the frankest manner—namely, if two straight lines in a plane are met by a third making the sum of the inner angles on one side of this third less than two right angles, then these two lines will meet on that side if sufficiently produced. Innumerable attempts were made to demonstrate this; but, at length, the efforts of Legendre and others made it pretty clear that this proposition could be deduced only from some other nearly equivalent. The least unsatisfactory assumption ever proposed was that of Playfair, that if of three unlimited straight lines lying in one plane two intersect, the third must cross one or both. It was at this point that Lobachevski cut the knot by supposing Euclid's postulate untrue, and showing that the result was a perfectly consistent system of geometry which may, for all we can yet observe, be the system of nature. All this time, Euclid's proof (elements, Bk. I., props. 16 and 17) of what substantially amounts to the proposition,

that the sum of the three angles of a triangle are *not greater* than two right angles, was regarded as perfect. It was not till 1854 that Riemann first discovered that, though accepted for two thousand years as conclusive (and it stands to-day unchanged in almost all the text-books), this pretended proof is really quite fallacious. It is plain that it is so, because it uses no premises not as true in the case of spherical as in that of plane triangles; and yet the conclusion drawn from those premises is known to be false of spherical triangles.

The truth is, that elementary geometry, instead of being the perfection of human reasoning, is riddled with fallacies, and is thoroughly unmathematical in its method of development. It has in some measure confused all mathematics, by leaving unnoticed most of the really fundamental propositions, while raising to an undue rank certain others almost arbitrarily selected. It leads young men into bad logical ways; and it causes pupil and teacher to think that whoever has difficulty with this sophisticated logic is wanting in aptitude for the apprehension of mathematics. The study of geometry ought to begin with the theory of perspective. Let a man be supposed to stand on an unbroken sandy plain. Let him fix a needle upon a post, and set up a plate of glass in a steady position, and draw a perspective picture upon the glass by placing his eye so as to bring the needle point over each point in the sand to be represented and marking it on the glass in the same line of sight. The horizon is where the lines of sight just skim the surface of the rounded earth. These lines of sight form a cone, and their perspective representation will be the section of this cone by the plane of the glass. But for simplicity let it be supposed that the earth is flat and indefinitely extended, so that the *plain* is also a *plane*, and an unbounded one. Then every straight line in the sand will have a straight line for its picture, for all the lines of sight from the needle-point to points in that straight line will lie in one plane; and this plane will cut the plane of the glass in a straight line.

Lobachevski and Riemann cast no manner of doubt upon the geometry of perspective, so far as this is confined to questions of incidence and coincidence. But when it comes to the measurement of distances and angles, their objections begin. According to the Euclidean notions, the infinitely distant parts of an unbounded plane would be represented in perspective by a straight horizon or vanishing line. But Lobachevski says we cannot be sure that this line would be straight, that may be it would be a hyperbola like the perspective of the terrestrial horizon; and, in fact, the straight line being only a special case of the hyperbola, it is proper to say that such is its form. Riemann, however, points out that we cannot even be sure there would be any such line at all, for we cannot be sure that space has any infinitely distant parts, since it may be that if we were to move off in any direction in a straight line, we might find that, after traversing a sufficient distance, we had got around to our starting-point again.

Prof. Halsted's translation (which, while our notice has been waiting, has reached, we are glad to see, a fourth edition) is excellent; his useful bibliography of non-Euclidean geometry was already well known. We could only wish there were a more copious appendix. The work of Lobachevski, though simple and convincing, is not what would now be considered a scientific presentation of the sub-

ject, and is open to a good deal of criticism. A new synthetic exposition is much needed, and might well accompany a collection of the contributions of Lobachevski, Bolzai, Riemann, Cayley, Klein, and Clifford.

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GEOMETRY NOT MATHEMATICS

Maxime Bôcher (1867-1918) was graduated A.B. from Harvard in 1888, and Ph.D. from the University of Göttingen in 1891. He began as an instructor of mathematics at Harvard in 1891, and in 1904 was given full professorship. He was a member of the American Mathematical Society (president, 1913-1914) and of the National Academy of Science.

To the Editor of The Nation:

Sir: A dissatisfaction with the method in which geometry is at present taught finds frequent expression, a case in point being the review of Mr. Halsted's translation of Lobachevsky in the last number of the *Nation*, and, by a peculiar coincidence, in the communication by Mr. Stille on "Science in America," in the same number. In the one case, the author desires to have geometry attacked by the beginner from the point of view of perspective; in the other, criticism is not aimed at the elementary presentation of the subject, but a spirit of discontent is visible which expresses itself in the demand that more students should go beyond the dull elements and breathe the purer air of the "modern geometry" of Jacob Steiner.

Does not the difficulty lie deeper than is suggested by either of these two writers? There seems to be a fundamental misconception of the nature of geometry in the minds not merely of the general public, but also of almost every teacher of the subject. "Mathematics is the science which draws necessary conclusions," so wrote the late Prof. Benjamin Peirce in 1870, and no clearer and more accurate definition could be imagined; and yet it has produced but little effect as yet on students of geometry. Geometry is the simplest of the natural sciences to which, owing to its simplicity, mathematics (*i. e.*, the methods of formal logic, either with or without the assistance of symbols) has been applied with such wonderful success, and which of late years has so richly repaid its debts to mathematics in the hands of Riemann, Clebsch, and others still living. We should laugh at the idea of teaching the mathematical theory of electricity before the student knows in a qualitative way what electricity is, either by a description of the phenomena or, better still, by actual laboratory contact with them. Why should not the same rule apply to geometry? Even then logic need not be entirely discarded, but let it assist the learner, not clog his progress. It is not a perfectly safe guide, as history tells us, even in the hands of a master, and the learner is almost as safe in trusting to his "common sense" as to his own unaided deductions.

The training of the logical faculty is, moreover, of far less vital importance than the development of the mind which results from this first real draught from the cup of science, which strengthens a healthy imagination, and should even yield an