each of which is susceptible of further mathematical treatment. It is, thus, a theory (for so mathematicians use the word theory) of great utility; and, like other utility-mathematics, is tedious, difficult, disagreeable, and unbeautiful. This is a circumstance which breeds many loathers of mathematics, because these disagreeable branches are taught first.

The present treatise is undoubtedly the best in our language upon this subject. Its only rival, that of Todhunter, always an unnecessarily dry book, is now pretty antiquated likewise. Mr. Byerly adheres to one point of view pretty consistently, exhibits the doctrine under its best aspect, and leads us into it by the easiest road. It is a branch which nobody but a practical mathematician will care for, and which every practical mathematician has to master.

When we turn from this book to Klein's lectures, we seem to be passing out from a tremendous, rattling factory, with its grimly earnest, unlovely economy, into the pure meadows with the really vastly greater, but infinitely calm, agencies of sunshine, breeze, and river. Here, in only a hundred pages, the moving impulses of modern mathematics are set forth in a way in the highest degree instructive and interesting to every mathematician, without any tax upon his energies. Felix Klein, we need hardly say, is generally considered as the most interesting, if not the greatest (certainly *not* in all respects), of living mathematicians. For such a hundred pages as these the mathematician may search in vain. The small compass renders the process of mathematical cogitation all the clearer, and strips it of details which in other books obscure it; and particularly of details of demonstration that are often wrongly taken to be the soul of mathematical thinking. Such a lesson as this book affords of the conduct of mathematical research the younger student (it is not for beginners) will not easily find. Those who know Klein need hardly be informed that the lectures range over a large part of recent mathematics. The following passage (in which we take the liberty twice to put *experience* in place of "conception") is interesting:

"We are forced to the opinion that our geometrical demonstrations have no absolute objective truth, but are true only for the present state of our knowledge. These demonstrations are always confined within the range of space experiences that are familiar to us; and we can never tell whether an enlarged experience may not lead to further possibilities that would have to be taken into account. From this point of view, we are led in geometry to a certain modesty, such as is always in place in the physical sciences."

Appended to the lectures are ten pages on the history of modern mathematics in Germany.

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CAJORI'S HISTORY OF MATHEMATICS

A History of Mathematics.

By Florian Cajori. Macmillan & Co. 1894.

CSP, identification: MS 1393 (draft). See also: Burks, *Bibliography*.

A brief sketch of the course of human activity in a way that has been of more avail than many a more showy trade in moulding men's daily lives and sentiments into what they are to-day, this book will attract the attention of every youthful student of any mathematical science or of engineering. For any man in such a line, who has the ambition to be more than the merest specialist, needs to be familiar with as much of the history of mathematics as can be crammed into one moderate volume. Others, students of humanity, to whom nothing can be more interesting and even entertaining than mathematical history, will find profit in this work unless it is too small for their purposes. We noticed a few years ago a volume on the same subject by Walter W. Rouse Ball, entertaining enough, but not a careful nor competent work. That contained perhaps 900,000 words; and the present compendium has, we estimate, about two-thirds of that number. It is too brief, but economizes space pretty severely.

The mathematics of the nineteenth century was treated by Ball not too seiously. His chapter on the subject, sooth to say, had an old-fashioned English flavor, and about as much thoroughness as one would expect in a collection of Lives of Eccentric Personages. Prof. Cajori, on the other hand, treats recent mathematics worthily, giving to it, as was proper, a quarter of his whole space. In this part at least his work is distinctly of a higher grade than Ball's and has evidently cost him original labor. While excessively condensed, as the fair proportions of his whole history required, it fulfils well enough the needs of students before the fascination of a deeper study of the history of thought has drawn them further into its vortex. We will not say an extraordinary, but a satisfactory, degree of discrimination and accuracy marks this section of the work.

Room could be made for this expanded account of the achievements of this century only by an extra turn somewhere upon the hydraulic press with which the rest has been baled. The part which seems to have been chosen to undergo the extremest compression is the period from Descartes to Euler. One cannot but regret this, for in some respects it is the most important period of all; and no other, certainly, calls more for those elucidations of the historian which show the inexperienced student in what the extraordinary advance of different steps in the progress of thought consisted. The main facts are given by Cajori, and often more correctly than by Ball; but there is no room to suggest reflections, and reflections make the worth of history. Take, for example, that truly gigantic reasoner, Fermat. One of his exploits was a method of ascertaining that value of a variable that renders the value of a given function the greatest or least in that vicinity—a method which, in the opinion of some of the greatest judges, constitutes Fermat the author of the differential calculus. Cajori states the matter so very briefly that the young reader will fail to perceive that the method is almost exactly the same as that pursued in the best modern treatises. The following are the steps of the process: (1) to the variable is given an increment which is assumed to be such that the value of the function before the variable is thus increased can be equated to the value after the increase; (2) by development and transposition, the increment is shown to be a factor of the whole equation; (3) the equation is divided by this increment, which will, in general, leave some parts of it independent of the value of the increment and some parts not so; (4) the increment is put equal to zero, and the value of the variable is then deduced. This is open to the objection that if the increment equals zero, we have no right to divide the equation by it. This objection is flanked in modern works by supposing the increment *is not* zero, but diminishes so as to have zero for its limit.

How very clever Fermat's method was is a thing the beginner needs to have pointed out. It was an idea sufficiently clever to change the whole face of the globe, with our daily mode of living and of philosophizing, to such an extent that the Egyptian under Khufu lived and thought more like a sixteenth-century man than the latter like us. For that all modern science has grown out of the germinal idea of the differential calculus we hold to be most surely true. The ἀγεωμέτρητος, as Plato used, it appears, to call the twaddler in thought, may laugh, not seeing how a prolific conception can assume a guise so humble. He would have a great idea essentially warm, picturesque, colored. He refuses to take seriously a pure intellect, whose forms are as applicable to a game of dominoes as to the struggles of world forces. We instance these as examples of the sort of reflection which, whether finally accepted or rejected, a good history of any wide branch of human thought ought to suggest, if not to discuss.

The above was not Fermat's greatest contribution to the art of reasoning. The greatest was the Fermatian inference, characterized by Prof. Cajori as "an inductive method," but that is quite to miss its essential peculiarity. We find Fermat's not publishing his work ascribed to an "uncommunicative disposition"; but this is thoroughly unjust.

We meet reiterated here for the thousandth time that tasteless objection to Pope's monumental couplet on Newton, to the effect that the important steps of discovery do not take the world by surprise, but were led up to so gradually as to be made almost unawares. It is the worst of German taste to criticise such a couplet because it does not accord with profound historical researches, so long as it expressed what seemed to ninety-nine out of a hundred of Newton's comtemporaries to be the truth. Moreover, the theory of intellectual development on which the objection proceeds is in silliest conflict with psychology and with history. No doubt a very large part of the progress of science is accomplished by industrious hammering away at plain jobs. But there is a part which cannot so be accomplished. To say no great strides are ever taken is in flat contradiction to the record. It is by no means as yet proved that even biological evolution always advances by almost imperceptible differences. It is a recognized principle of modern biology that the nature of the history of the evolution of the race may be judged from that of the evolution of the individual. Now, besides the daily growth of individuals, we see them passing through wondrous and sudden transformations. Of course, there is no absolute breach of continuity, but for that no man contends.

Be the matter as it may in the natural world, the development of thought being open to study, from both outside and inside, from the world's history and from the individual's experience, men of sense and observation ought by this time to have reached some settlement of opinion on this point; and so, perhaps, they have. Doubtless there are men who are never surprised, but whether the wise count them among their number is not so clear. At any rate, he who attempts to expound the history of science, and à fortiori the history of mathematics, without recognizing that great, startling, and revolutionary discoveries from time to time get made, will have but a wrenched, unjointed, and enfeebled account of it to offer to his readers. We sometimes hear such facts as the knowledge of the intoxicating property of ether before Morton, of a few observations of spectra before Bunsen and Kirchhoff, and the like, put forward to show that no great discoveries are startling. But these instances, if they prove anything, prove just the reverse. They show that, as Whewell said, observations are nothing to a mind unstored with appropriate ideas, and thus emphasize the importance of originality of mind, of which novelty is the product.

Prof. Cajori furnishes a list of one hundred works on the history of mathematics that will be very welcome. These are the works of which he has made use. We notice he has not had the advantage of using Boncampagni's *Bulletino*, nor the *Bibliotheca Mathematica*, as he would have done if he had worked in New York, where it is for the interest of the public that men devoted to the history of mathematics should be stationed. The Astor Library is particularly rich is this direction. We do not find on his list Weissenborn's book on the introduction of the Arabic numerals by Gerbert (1892). We remark, by the way, that one class of events to which some attention might well have been paid is that of the establishment of the several journals of mathematics.

Prof. Cajori's transliteration of Arabic is irregular and puzzling. Speaking of the celebrated algebra of Muhammed ibn Mūsā, the Khivan, he says: "The name of the author, *Hovarezmi*, has passed into *Algoritmi*, from which comes our modern word *algorithm*." The appellation he refers to as the "name of the author" is *al-Khwarizmy*, meaning 'the native of Chorasmia, or Khiva.' He writes *al-jebr* (algebra) in the form "aldshebr," putting *dsh* for the letter *jim*; but for the native of Khojend, or the Khojendy, he writes *Al Hogendi*, using a soft *g* for the same purpose; and in writing *Abul Gud* and *Abu Gafar Al Hazin* (where "Gafar" is the Ja'afer so familiar in the Arabian Nights) he makes a hard *g* serve. He also writes *Ulug beg* where the English *g* stands in place of a second and a third letter of the Arabic alphabet (the last being the Persian gāf). This cognomen of Muhammed Tarāghāy is best written Ulugh Beyg. It means "the great lord." We read: "Creditable work was donce by *Fahri des Al Karhi*, who lived at the beginning of the eleventh century." Now, there was an Abū Ghālib at that time, who was surnamed Fakhr al mulk, "glory of the realm." After this high personage, the mathematician Abū-Bekr-Muhammed-ibn-alhusain, called Al-Karkhy, named a book 'Al-Fakhry,' where the final *y* is equivalent to our termination *-ian*. It amounted to a dedication. As for Cajori's "*des*," we can see in it nothing but the German genitive article.

But these are trifling faults. What we have a right to expect in such a handbook is an agreeable narrative of the most material events in the history of mathematics, and this Prof. Cajori incontestably supplies. The book was much wanted.