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THE NATURE OF LOGICAL AND MATHEMATICAL THOUGHT.

INTRODUCTORY.

HUMAN thought is dominated by methods based upon a principle which in its various applications is called reason, and the total set of the rules of reason is called logic. In the same way all computations, and further all space-conceptions, measurements of distance and of direction, depend upon a science which has much in common with logic and has received the name "knowledge-lore" or "mathematics."

Experience has shown that the accomplishments of both logic and mathematics are most marvelous. They are the woof of the web in the fabric of all the sciences, and human civilization is their most palpable product. In fact we may say that man himself, especially the scientific thinker, is nothing but reason (viz., logic and mathematics) incarnate. All that distinguishes man from brute creation consists in his ability to think with definite methods, to be logical and exact in measuring and counting.

Now it is strange that the nature of man's rationality is by no means universally recognized. Opinions vary greatly concerning its foundation and its origin, and this divergence has come out most plainly in a new development of mathematical thought which has produced peculiar systems of mathematics differing from the traditional Euclidean system. There is still missing, however, a new system of logic which would be contradictory to the logic of Aristotle.

The revolution against the old views began with an attack on the axiom of parallel lines; and the idea that through any point C, there ought to be one and only one straight line parallel to a given straight line AB, has been set aside to make room for a higher and more general mathematics, a pangeometry, where the Euclidean assumption would be only one special case among other possibilities. Other new systems have indeed been developed in which Euclid's parallel postulate is set aside as unproven, and as a result the view has been commonly accepted that other non-Euclidean geometries are possible.

Mathematics is at present dominated by a tendency which may be called experimentalism. The mathematician hungers for facts, for a basis in the realm of concrete sense-experience. He envies his brethren the naturalists, whose methods since the days of Darwin have enjoyed an enormous boom. He has been living for centuries in a domain of pure thought, and he wants now to stand on the ground of actuality. The natural sciences have attained wonderful results in the shape of inventions and discoveries, and some mathematicians feel that they are left behind in the race and so they are making vigorous efforts to emulate the naturalist method of investigation.

This hankering for facts in the domain of mathematics is in our opinion an aberration. Mathematics is a creation of pure thought. It is built up in the domain of anyness—a product of abstraction. Questions as to the nature of actual (i. e., objective) space, whether it be Euclidean or non-Euclidean, homaloidal or curved, homogeneous or heterogeneous, three-, four-, or n-dimensional, are all beside the mark. Mathematicians of great repute, who at the same time are masters of all the details of their science, have raised questions such as these: "Will not a straight line finally, after billions of miles (or perhaps at a distance of billions of light years) return into itself?" or "May not the sum of the angles of a plane triangle, if only measured in cosmic proportions sweeping through the stellar heavens, prove to be a little more or a little less than 180 degrees?" or, "Are the opposite angles in a parallelogram really equal?" or "Is objective space such as we think it or is it different? Is space Euclidean or non-Euclidean? Is it tri- or four- or many-dimensional?" All these and kindred problems prove that those who propose them,—I say so with all deference to their learnedness and yet with perfect assurance—do not understand anything of the foundations of mathematics.

According to my conception of mathematics, we have created the plane, and in plane geometry the straight line is straight and remains straight into infinity, the right angle is a right angle wherever it may be constructed in a plane, and the angles of a plane triangle measure exactly 180 degrees, nothing more, nothing less. There is no approximation; everything is exactly so. Such is the nature of mathematical thought which, in this respect, is different from the facts of the natural sciences. In the natural sciences our observations and measurements are never perfectly exact; they are always approximations.

The natural sciences deal with particulars, and generalized statements have been gained by induction from an observation of several or many particular experiences. But logic and mathematics are sciences of pure form and their productions are mental constructions which are rigidly and unequivocally determined, and there is no approximation about their truth. I may add here that as there are no mathematical planes and lines so there are no syllogisms in the objective world of fact, but there are uniformities for the tracing of which logical rules are serviceable, and in the domain of logic the syllogisms are as rigid as are the propositions of Euclid in plane geometry.

To the new-fangled non-Euclideans and to adherents of the New-Science conception this statement may appear antiquated and old-fashioned, but a close inspection will prove that science still stands on the old foundations, and though in the course of modern development new and broader viewpoints have been gained, science will after all be found to remain on the Rock of Ages, on that irrefragable consistency of natural events which can be formulated in the so-called natural laws and finds its noblest development in the rationality of the human mind, viz., in those eternalities which are ultimately nothing but the consistency of thinking, the consistency of doing, the consistency of being. The author has published two books on this most important problem, *Kant's Prolegomena* (being a critical discussion of the Kantian solution) and *The Foundations of Mathematics*; and he wishes here to present a brief recapitulation of his views and add some comments on conceptions which differ from his own.

THE AUTHOR'S POSITION.

The belief in a consistency of existence is first a mere faith, based upon an instinctive apprehension of law underlying all regularities; but this faith proves the more reliable the deeper we penetrate into the nature of being.

A condition of uniformities which admits a possibility of formulating them in natural laws is called in German, *Gesetzmässigkeit*, and this has as yet no equivalent in English. We propose to call it "lawdeterminedness" or simply "lawdom,"* and would define lawdom as a state of things in which all events take place according to general rules,

* The word "lawdom" is formed in analogy to kingdom, freedom, wisdom, Christendom. *Dom* is derived from the same root as *doom*, "judgment," and means in kingdom the dominion of a king; in wisdom, the prevalence of the wise; in freedom, the sway of the free; and in lawdom, it means a condition determined by law.

viz., the laws of nature. In this sense we say that the consistency of natural phenomena manifests itself as lawdom.

Every contradiction is a problem and every solution of a problem becomes a renewed justification of our belief in the consistency of existence. This belief appears for a time as a divine revelation and finally becomes the assured result of science. If there were no consistency there would be no science, reason would be a mere coincidence of haphazard regularities, and a trust in the efficiency of reason should be branded as a vagary of deluded dreamers.

The very existence of reason is an evidence that the universe is consistent throughout, and human reason is an instinctive comprehension of this most remarkable nature of existence, while science is simply the methodical application of reason.

This resumé sounds very simple, yet sometimes it is difficult to state and comprehend simple truths. We shall have to grant that simple truths stand in need of elucidation, for in the infinite manifoldness of actual existence they are rendered quite complex, and thus it happens that great thinkers encounter many difficulties which can be surmounted only by a most scrupulous exactness.

While experience and experiment can not settle the problems as to the nature of mathematical space, we must grant that there is one great truth in the tendency of modern mathematics. It is this, that mathematics is not absolutely independent of experience. Though mathematics is a purely mental construction, the method of its construction is derived from experience. In other words, though mathematics is, in the terminology of Kant, *a priori*, our *modus operandi* is a function which we have procured by abstraction from our activity evinced in the domain of the *a posteriori*. We cancel in thought everything particular which comprises all things concrete, be they of matter or energy, and retain only our mental faculty of doing something, including a field of action implied by the possibility of moving about. This field of action with its absolute absence of all particularity is characterized by generality. In other words, it is the domain of anyness.

Accordingly we do not start in mathematics with nothing, nor do we go about our business blindly. In arithmetic we operate by taking a step and repeating it again and again. Thus we posit a unit, then we proceed to posit another unit and another, and each unit is the same as all the rest. We count them and operate with their sums. Such is arithmetic or the science of numbers.

When we bear in mind that mathematics is a mental construction we will readily understand that sums in arithmetic are products of synthesis. Every number is the result of an addition, and addition is no mere analysis of the idea of number; it partakes of the synthetic character and becomes possible only through the procedure of positing new units and summing up the total result.

Kant was astonished to find that even the most simple arithmetical calculation (such as $8+5=13$) was not the result of a mere analysis of the numbers implied, but was of a synthetic nature. Analytical judgments do not teach us new truths; they only render the ideas we have clearer and more definite, while synthetic operations increase our stock of knowledge. This puzzled him, for according to his nomenclature all mathematical, arithmetical and logical propositions were *a priori*, and all *a priori* propositions were quite commonly (though erroneously) assumed to be purely analytic. So he came to the conclusion that man's faculty of making *a priori* constructions constituted in itself a source of positive knowledge,—of knowledge that could be increased and amplified without resorting to sense-experience.[1]

By *a priori* knowledge Kant understands all that knowledge which is presupposed in experience of any kind. When the chemist analyses some compound and finds in his retorts 87% of its mass, he concludes that he lost 13%; he does not assume that during the process 100 particles have shrunk into 87, or that 13% have vanished into nothing. They are lost to him but have not been annihilated. In the same way all mathematical and logical propositions are relied upon. They are trusted above all experience, and if we make an experiment the result of which contradicts them (or seems to contradict them), we doubt our observations and distrust our experiment. We seek the fault in our notion of the facts in question, not in the principles of reason. We may distrust our calculations and our arguments, but we never doubt the reliability of mathematics and logic. If an astronomer watches a comet, and determines three stations of its course by observation, he can map out a curve which is analogous to its path of motion, and in the same way all the formal sciences furnish us with a key that will unlock to us the mysteries of objective existence. This state of things, the agreement of our purely formal thought-construction with the laws of nature, is a most wonderful coincidence and it puzzled Kant to such an extent as to make of him an idealist, but the problem is solved if we bear in mind the "anyness" which characterizes our purely formal constructions. If consistency dominates both objective existence and our thought, both will be analogous.

How do we produce this anyness?

In geometry we begin with mapping out our field of operation. First we ignore everything actual or concrete; both matter and force are treated as if they were nonexistent and all that is left is motility. We can move in

[1] I use the term sense-experience on purpose so as to limit the meaning of the term and to avoid the mistake resulting from the looseness of its use in Kantian nomenclature. Cf. the author's *Fundamental Problems*, pp. 26 ff., especially 30.

any direction and everywhere without end. Suppose we spread out in all directions at once by swelling up, or by spreading like light from a source of luminescence, we would cover the entire possibility of our scope of motion. In such a spread of motion we call a path of greatest intensity corresponding to a ray of light a straight line. Now we cut space in two and call the boundary between the two halves a surface. If the cut has been made evenly, which means by a ray and along a ray, i. e., by a straight line, which is a line that follows the path of greatest intensity,[2] we can flop the surface upon itself and we call it "a plane."

As a visible representation of the plane we use a sheet of paper which when folded upon itself produces the straight line. We use the folded sheet as a ruler and operate with it. We lay down units of length (feet and inches, or meters and centimeters) for the sake of measuring lines. Then we draw straight lines in different directions and make them intersect. Their products are angles. We make three lines intersect and call the figure thus created a triangle.

Further on we fold the creased paper upon itself and name the corners right angles. The plane, the straight line, the right angle are boundary conceptions which are useful because they are unique. There are innumerable curves, but only one straight line; there are innumerable obtuse and acute angles, but only one right angle; and thus these boundaries, these products of halving, will serve us as standards of reference.

Our next step is the creation of a curve that by its simplicity would possess the advantage of uniqueness. So, we draw a circle on our doubly folded sheet of paper from the point where the two creases meet. Following historical tradition which can be traced back to the sages of

[2] Compare *Foundations of Mathematics*, pp. 57-58.

ancient Babylon, we divide the whole circle into 360 degrees and we may remember here how their mathematical instinct was guided and influenced by some facts of observation. They rounded off the number of days from 365 to 360,[3] and divided the course of the sun on the ecliptic into twelve mansions of 30 degrees each, corresponding to 12 double hours per day.

The next step in geometrical constructions will be the transfer of angles and the drawing of two straight lines running in the same direction. We call them parallels. When two parallels are crossed by a third straight line, we investigate the nature of the eight angles thus produced.

We continue to operate by setting ourselves a series of tasks, and in doing so we can follow Euclid's propositions in their regular order by dealing with three intersecting lines and then with the circle and other figures. In this way we build up plane geometry without axioms or assumptions through our own operations, and we remain conscious of the method by which we came into possession of the straight line, the right angle, the parallel, etc.

There is nothing actual about our operations. All our achievements are purely mental; they lack concrete reality. There is no matter, no force in our constructions, and yet they are not nothing. The path of our motion is a line, and where two lines cross we have a point. A point is no concrete thing, yet it is not a nonentity; it is a locus in the field of our motion, a spot the position of which is definitely determined on either of the crossed lines. We do not find a plane anywhere in actual life, we construct it; and in the same sense straight lines and right angles are the products of our construction.

[3] The difference was made up every sixth year by the introduction of an intercalary month—the month of the raven, and it is noteworthy that the number 13 as well as the symbol of the thirteenth monh, the raven, have remained omens of ill luck to this day.

Nothing proves so well that our space-conception is in Kant's sense *a priori* as the possibility of non-Euclidean geometries. There is only one rule to guide our operations, consistency, and since particularity of any kind has been banished, the same operation will always and everywhere produce the same result. We operate in an absolutely empty field and our constructions are solely determined by the nature of our operations. All we have to do is to note the consequences of our transactions.

We might have constructed another field for our operations, for instance, the surface of a globe; and if we had done so from the start, our products would have been different. Straight lines would have become impossible and lines analogous to Euclidean straight lines would be largest circles. They are "shortest" or "straightest" lines; not truly straight in the Euclidean sense, they are the straightest lines possible. While two straight lines in the plane never enclose a space, two straightest lines on the sphere always enclose a space; and while the former intersect in one point (or if they are parallel not at all), the latter always intersect at two points and these two points are antipodal.

The construction of spherical geometry is quite simple and it is as easily pictured in visible figures as plane geometry; but there are other geometries possible, less simple and more difficult to describe or to render representable. Each one of them possesses its own characteristics and theoretically considered all of them are equally legitimate. They are all mental constructions. They are all based upon the principle of consistency and obey the general laws of logic, for if they did not recognize consistency, our operations would end in chaos.

The systems of Euclid, of Bolyai, of Lobatchevsky and others, including 4- or *n*-dimensional manifoldnesses, are *a priori* on the same footing. The difference comes in

when they are applied to practical purposes, and here the Euclideans have after all the advantage. The non-Euclideans make up for it by an enthusiasm as strong as the zeal of religious devotees which on the one hand deserves our admiration while on the other it has a humorous aspect.

That logic and mathematics come from the same root must have been felt by Euclid and his school, for what they call "common notions" are formulations of logical principles, while the description of the characteristics of space are laid down in the definitions and postulates. But the significance of the kinship between these two sciences, it appears, was first felt by Kant who may have been guided in this by his great contemporary Lambert.

Since the time of Kant, both logicians and mathematicians have felt the need of investigating the nature of thought-operations and of broadening the concepts of logic in a similar way[1] as the metageometricians endeavor to construct a pangeometry which would be independent of our conception of Euclidean space. The first classical work which broadened the traditional logic was written by George Boole under the title *The Laws of Thought*, and since then logicians have felt the insufficiency of Aristotle's logic and the need of deciphering the nature of thought in its operations. They attempted to transfer the accomplishments of mathematics upon logic, and to exhibit the function of reason in formulas, or in graphic presentations, or in algebraic notations. Workers in this line are Ernst Schroeder, Charles S. S. Peirce, Giuseppe Peano, Bertrand Russell, and Louis Couturat. It is a new branch of scientific endeavor and we may expect results of great interest, yea even of far-reaching importance.

The writer's opinion is that labors of this kind constituting the new mathematics and the new logic are quite legitimate. They will widen our horizon but they do not (and never will) reverse, antiquate, or abolish the assured accomplishments of the past. Neither Bolyai nor Lobatchevsky upsets Euclid and none of the modern logicians will ever set aside Aristotle.

NON-ARISTOTELIAN LOGIC.

Now, it is possible to imagine a fairy-tale world where our scientific conception of cause and effect could be crossed by a causation of miracle. In such a world the magician's word would be endowed with an energy unknown in physics, but it would remain a world governed by law, and the rule of consistency would not be upset. Every effort would presuppose a cause and causation would still be dominated by law. The purely formal rules of Aristotelian logic would not be upset thereby. The mill remains the same even if the grist is changed. We would have law-determinedness or lawdom in both worlds. The forces and materials would be different but not the consistency of the concatenation of events.

Aristotelian logic is incomplete and insufficient. It treats only the most simple relations and does not cover the more complicated cases of thinking, but so far as it goes it is without fault. If we grant that all men are mortal and that Caius is a man, we must make the conclusion that Caius is mortal—otherwise he would not be a man but some immortal being, and this would upset the principle of consistency.

We might assume that there are no uniformities in nature, or that all rules have exceptions, or that the uniformities are mere approximations, in which case we would have a world of haphazard happenings. But that would never upset either *Barbara* or *celarent*, or any other rule of pure logic. The items of actual existence would not be classifiable, but the Aristotelian method would not thereby become wrong.

Some time ago I made the following comment on the nature of logic in the *Primer of Philosophy*, (p. 109):

"Mathematicians with great ingenuity have invented various kinds of mathematics. They have shown that Euclidean geometry is but one actual case among many possible instances. Space might be curved, it might be more than three-dimensional. But no one has yet been bold enough to propound a theory of curved reason.

"And why should there not as well exist a curved logic as a mathematics of curved space? A curved logic would be a very original innovation for which no patent has yet been applied for. What a splendid opportunity to acquire Riemann's fame in the domain of logic!"

Now it happens that my friend, Mr. Francis C. Russell of Chicago, received a letter on sundry topics of modern logic from Mr. Charles S. S. Peirce, known as one of the most prominent logicians, and it contains a most interesting passage which sounds like an answer to this challenge of mine. With the permission of the writer I quote it in this connection:

"Before I took up the general study of relatives, I made some investigation into the consequences of supposing the laws of logic to be different from what they are. It was a sort of non-Aristotelian logic, in the sense in which we speak of non-Euclidean geometry. Some of the developments were somewhat interesting, but not sufficiently so to induce me to publish them. The general idea was, of course, obvious to anybody of sufficient grasp of logical analysis to see that logic reposes upon certain positive facts, and is not mere formalism. Another writer afterward suggested such a false logic, as if it were the wildest lunacy, instead of being a plain and natural hypothesis worth looking into [notwithstanding its falsity]."*

I begin to think that Mr. Charles S. S. Peirce understands something else by Aristotelian logic than I do.

* In giving his consent to publish this extract from his letter, Mr. Charles S. S. Peirce sends an additional explanation which is published on page 158 of the present number.

The world has seen many new inventions. Over the telephone we can talk at almost unlimited distances, and some of our contemporaries fly like birds through the air. Radium has been discovered which is often assumed with a certain show of plausibility to upset the laws of physics, but the invention of non-Aristotelian logic would cap the climax. We make bold to prophesy that the non-Aristotelian logic will abolish Aristotle as little as the non-Euclideans have antiquated Euclid. If it comes it will, if it be sound, give us new viewpoints, but it will not abolish one iota of the well-established truths of the old logic. Of course, a non-Aristotelian logic would be "worth looking into," even if it were a vain attempt. *Nous verrons.*

PROFESSOR BERTRAND RUSSELL'S VIEWS.

Since the publication of my two books on this subject, *Kant's Prolegomena* and *The Foundations of Mathematics,* I came across an article by one of the most famous mathematicians of our time, Professor Bertrand Russell of Cambridge, England, a scholar of great erudition and author of many valuable books, among which is an excellent book on *The Foundations of Geometry*. If Professor James or his pragmatist adherents speak of Euclid as superseded and no longer true, they are not to be taken seriously, and there is no need of refuting them; but the case is different when mathematicians of standing make similar declarations. Professor Russell's article on "Recent Work on the Principles of Mathematics," published in the *International Monthly,* is bewildering to me. The very style and presentation of the subject is fascinating, perhaps because the arguments seem paradoxical. At any rate the author's prominence has caused me to reconsider my own position, but I can only say that in spite of his unquestioned authority I cling to my own views. All I can do is to contrast his ideas with my own, and for the sake of fairness I will

quote extensively from his essay so as to let him present his views in his own words.

I shall begin with a quotation which I heartily endorse. Professor Russell says (pp. 84-85):

"Logic, broadly speaking, is distinguished by the fact that its propositions can be put into a form in which they apply to anything whatever. All pure mathematics—arithmetic, analysis, and geometry—is built up by combinations of the primitive ideas of logic, and its propositions are deduced from general axioms of logic, such as the syllogism and the other rules of inference. And this is no longer a dream or an aspiration. On the contrary, over the greater and more difficult part of the domain of mathematics, it has been already accomplished; in the few remaining cases, there is no special difficulty, and it is now being rapidly achieved. Philosophers have disputed for ages whether such deduction was possible; mathematicians have sat down and made the deduction. For the philosophers there is now nothing left but graceful acknowledgements."

My mode of thinking has complied with the demand. I would replace the expression "axioms of logic" by "the principle of consistency," but otherwise I would feel in perfect agreement with Professor Russell, if his article did not abound in many other statements which appear to me irreconcilable with this unequivocal and simple description of the situation.

The reader will notice that Professor Russell is rather hard on philosophers, but it can not be denied that philosophers, at least many men who have gained fame under that name, have unduly slighted mathematics. It is strange, though perhaps natural, that mathematicians like Schroeder and Peano have distinguished themselves in the construction of an algebra of logic. Furthermore there are a number of modern mathematicians, inspired by the broader and more philosophical conceptions of mathematical notions, who have advanced their science by taking new view-points. Professor Russell mentions three great Germans, Weierstrass, Dedekind and Cantor, whose merits are indubitable. Other names might have been added, such as Clebsch, Grassmann, Fuchs, Klein, Lindemann and Staudt, but I fail to see that any one of them has tried to solve or claims to have solved the philosophical problem of the foundation of mathematics. The great drift of their labors, so far as I can judge, is, with the exception of the work of Grassmann, purely mathematical.

I grant that Euclid has his faults, but I believe that his mistakes can be remedied. I also grant that "he is not an easy author and terribly long winded." I deem his proofs tiresome with the monotonous refrain, *Q. E. D.*, and I would replace his method as suggested above by changing the doctrinary style of propositions into the accomplishment of tasks. But for all that, Euclidean geometry remains classical, and I can not understand Professor Russell's harsh verdict when he says (p. 100):

"It is nothing less than a scandal that he should still be taught to boys in England. A book should have either intelligibility or correctness; to combine the two is impossible, but to lack both is to be unworthy of such a place as Euclid has occupied in education."

I do not agree with Professor Russell that "to combine the two," (viz., intelligibility and correctness) is "impossible." If that were so we would land in mysticism.

Here is another passage on Euclid. Professor Russell says (p. 98):

"It has gradually appeared, by the increase of non-Euclidean systems, that geometry throws no more light upon the nature of space than arithmetic throws upon the population of the United States."

True, the formal sciences never supply us with facts; but they offer us a method of dealing with facts, and that is better. Professor Russell continues:

"Geometry is a whole collection of deductive sciences based on a corresponding collection of sets of axioms. One set of axioms is Euclid's; other equally good sets of axioms lead to other results.

Whether Euclid's axioms are true, is a question as to which the pure mathematician is indifferent; and what is more, it is a question which it is theoretically impossible to answer with certainty in the affirmative. It might possibly be shown, by very careful measurements, that Euclid's axioms are false; but no measurements could ever assure us (owing to the errors of observation) that they are exactly true. Thus the geometer leaves to the man of science to decide, as best he may, what axioms are most nearly true in the actual world."

Since Euclid's geometry consists of constructions of pure thought, since there are no points, lines, surfaces, planes, etc., in the objective world, it is obviously impossible to test the truth of Euclidean propositions by actual measurement. Professor Russell does not define his conception of truth. We would say that a Euclidean proposition is true when it is an adequate or correct description of the results of a construction. The question is not, what axioms are most nearly true in the actual world, but which geometry is most serviceable in calculating the relations that obtain in the actual world.

Professor Russell frequently indulges in mystifications. He says (p. 84):

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate."

All this is *ingeniosius quam verius*. Statements can easily assume a paradoxical form when they are based upon an inaccuracy of terms. Mathematical propositions do not describe realities, but, because lines and planes are not real, we can not say that what mathematics teaches is "not true." Nor is it fair to define mathematics as "the subject in which we never know what we are talking about."

I understand that Professor Russell bases his view upon the method of some Italian mathematicians who avoid a discussion of the foundation of mathematics by the use of a conditional "if." They start their proposition by saying, "If I do this, the result will be such and such." The "if" sentence is purely hypothetical and they do not trouble about it, but if it be allowed to stand the result can not be denied. Professor Russell explains the situation thus:

> "Pure mathematics consists entirely of asseverations to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true and not to mention what the anything is, of which it is supposed to be true. Both these points would belong to applied mathematics. We start, in pure mathematics, from certain rules of inference, by which we can infer that *if* one proposition is true, then so is some other proposition. These rules of inference constitute the principles of formal logic. We then take any hypothesis that seems assuring, and deduce its consequences. *If* our hypothesis is about *anything*, and not about some one or more particular things, then our deductions constitute mathematics."

We may grant that "the rules of inference constitute the principles of formal logic." But why should it be "essential not to discuss whether the first proposition is really true?" I propose to avoid the vicious "if" which leaves the entire science of mathematics in the air, and to dig down to the bottom rock of our mode of thought and build the foundation that is needed for the superstructure of this noblest and loftiest of all the sciences.

If my conception of mathematics is true we do not need in geometry "a certain number of primitive ideas, supposed incapable of definition and a certain number of primitive propositions or axioms, supposed to be incapable of proof."[4] We remove every trace of particularity and build upon the abstract idea of "anyness" a universe of pure thought which will serve as a model for any possible formation, fictitious or real.

[4] *Ibid.*, p. 84.

Professor Russell lays much stress on the symbolic nature of modern logic, and I grant that the significance of symbolism can not be overrated. I would insist that language of any kind, yea even sense-perceptions, are symbolic, and the very nature of thought is symbolism. Sense-impressions change into sensations and sensations become perceptions solely through becoming symbolic. As soon as a sense-impression of a definite kind has come to represent some fact, an event or an object that causes it, then the sense-perception stands for or symbolizes the fact sensed. This is the origin of thought, and we have defined the soul as "a system of sentient symbols." Professor Russell apparently uses the term "symbol" in the more limited sense of an algebraic symbol. He says (p. 85):

> "People have discovered how to make reasoning symbolic, as it is in Algebra, so that deductions are effected by mathematical rules."

Algebraic symbols have the great advantage over language that they are definite and rigid. Language suffers from the fault of being vague. The use of our speech is incredibly loose and even the most common words, such as "to be," "to have," "we," "you," etc. possess several shades of meaning. This is not so in algebra and so logicians hope to overcome the looseness of reasoning in language by the employment of symbols which are as rigidly defined as the algebraic terms. The invention of such terms and of their mode of operation is a difficult task, and it would require a good deal of concentration of thought for any one to familiarize himself with a system of such an algebra of logic; but the gain is rich when we consider that thought acquires thereby the virtue of mathematical exactness. The trouble so far has been that there has been too little cooperation among logicians and almost every one of them invents symbols of his own.

Professor Russell's love of paradox appears in his exposition of the importance of symbolism. He says (pp. 85-86):

"It is not easy for the lay mind to realize the importance of symbolism in discussing the foundations of mathematics, and the explanation may perhaps seem strangely paradoxical. The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy of correctness. Hence we invent some new and difficult symbolism, in which nothing seems obvious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical."

We would not say that "symbolism is useful because it makes things difficult," but because it makes thought exact, and further, though it will prove difficult in the beginning, it will make exact thinking easy. It will show in a formula the machinery of thought and thus will render the process of thinking intelligible. In the same way a beginner in algebra may deem this mode of computation hard, but as soon as he has mastered its principles he will be enabled thereby to solve difficult problems without great exertion.

One of Professor Russell's observations is very good, though again stated in such a way as to make its truth appear in a paradoxical light. We must be on our guard against statements that appeal to us as obvious. The records of the history of philosophy and of religious dogma contain many flagrant instances of ideas deemed to be innate and of truths supposedly so obvious that it was claimed they did not stand in need of any proof. Professor Russell says (p. 86):

"The proof of self-evident propositions may seem, to the uninitiated, a somewhat frivolous occupation. To this we might reply that it is often by no means self-evident that one obvious proposition follows from another obvious proposition; so that we are really discovering new truths when we prove what is evident by a method which is not evident. But a more interesting retort is, that since people have tried to prove obvious propositions, they have found that many of them are false. Self-evidence is often a mere will-o'-the-wisp, which is sure to lead us astray if we take it as our guide."

When Professor Russell speaks of " a method which is not self-evident" I understand him to mean a method which must first prove its right of existence.

The mathematician should banish from his science any proposition which can show no other title than the claim of self-evidence. For this reason I have endeavored to do away with axioms and to build up mathematics without resorting to assumptions, self-evident statements, or asseverations of any kind. I wish Professor Russell would not describe mathematics as consisting of "asseverations"; the very idea is jarring on my conception of the nature of mathematics.

Among modern mathematicians Professor Peano has distinguished himself by an application of the algebraic method to mathematics in general, and Professor Russell looks up to him as a leader. He says (pp. 86-87):

"The great master of the art of formal reasoning, among the men of our day, is an Italian, Professor Peano, of the University of Turin. He has reduced the greater part of mathematics (and he or his followers will, in time, have reduced the whole) to strict symbolic form, in which there are no words at all. In the ordinary mathematical book, there are no doubt fewer words than most readers would wish. Still, little phrases occur, such as *therefore, let us assume, consider,* or *hence it follows.* All these, however, are a concession, and are swept away by Professor Peano. For instance, if we wish to learn the whole of arithmetic, algebra, the calculus, and indeed all that is usually called pure mathematics (except geometry), we must start with a dictionary of three words. One symbol stands for *zero,* another for *number,* and a third for *next after.* What these ideas mean, it is necessary to know if you wish to become an arithmetician. But after symbols have been invented for these three ideas, not another word is required in the whole development. All future symbols are symbolically explained by means of these three. Even these three can be explained by means of the notions of *relation* and *class*; but this requires the logic of relations, which Professor Peano has never taken up."

Further down on page 99 Professor Russell says:

"One great advance, from the point of view of correctness, has been made by introducing points as they are required, and not starting, as was formerly done, by assuming the whole of space. This method is due partly to Peano, partly to another Italian named Fano. To those unaccustomed to it, it has an air of somewhat wilful pedantry. In this way, we begin with the following axioms: (1) There is a class of entities called *points*. (2) There is at least one point. (3) If *a* be a point, there is at least one other point besides *a*. Then we bring in the straight line joining two points, and begin again with (4) namely, on the straight line joining *a* and *b*, there is at least one other point besides *a* and *b*. (5) There is at least one point not on the line *ab*. And so we go on, till we have the means of obtaining as many points as we require. But the word *space*, as Peano humorously remarks, is one for which geometry has no use at all."

There is no need of using the word "space," but is not the idea of space of some kind presupposed in the notion of a line, or even in the notion of a point? What is a point except a spot in space? Professor Russell must excuse me for finding Professor Fano's method of avoiding the difficulty comical. He starts "There is a class of entities called *points*. There is at least one point. If *a* be a point, there is at least one other point besides *a*." This is all very nice and begins like a fairy-tale, "Once upon a time." He rushes these statements upon us with an unmitigated abruptness which is truly naive. He has points, lines, distances, directions, but knows nothing of space. The very word "space" is abolished! Such are "the rigid methods employed by modern geometers" that "have deposed Euclid from his pinnacle of correctness"!

I feel strongly inclined to enter into Professor Russell's discussion of Zeno's problem, but space forbids. It would take an essay by itself, but a few comments on the subject may be permitted. Professor Russell presents the issues so interestingly that I wish I could read the whole exposition to my readers. A sample will prove that this is not mere courtesy. Professor Russell speaks of the infinitesimal as follows (pp. 89-90):

"The infinitesimal played formerly a great part in mathematics. It was introduced by the Greeks, who regarded a circle as differing infinitesimally from a polygon with a very large number of very small equal sides. It gradually grew in importance, until, when Leibnitz invented the infinitesimal calculus, it seemed to become the fundamental notion of all higher mathematics. Carlyle tells, in his *Frederick the Great*, how Leibnitz used to discourse to Queen Sophia Charlotte of Prussia concerning the infinitely little, and how she would reply that on that subject she needed no instruction—the behavior of courtiers had made her thoroughly familiar with it. But philosophers and mathematicians—who for the most part had less acquaintance with courts—continued to discuss this topic, though without making any advance. The calculus required continuity, and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be. It was plainly not quite zero, because a sufficiently large number of infinitesimals, added together, were seen to make up a finite whole. But nobody could point out any fraction which was not zero, and yet not finite. Thus there was a deadlock."

So far as I know, mathematicians have never taken this deadlock seriously, for they know that the infinitesimal is a fiction. There are no infinitesimals in the objective world, and in the ideal realm of mathematics it is an attempt to represent a continuum under the aspect of discrete units, which is necessary for the purpose of computation.

We have stated above that all thought is symbolic, and the method of thought depends upon the symbols we employ. There are two possibilities; we can proceed either in a path of uninterrupted motion or we may cover the ground in steps. The former method is geometrical, the latter arithmetical. The former is a continuous progress, the latter an advance in counting units. The former has the advantage of presenting outlines of pictures in their totality as images; it is qualitative. The latter sums up numbers fit for use in computations; it is quantitative. Now it so happens that sometimes we need one and sometimes the other. A geometrical curve is a continuum, and so if we wish to calculate it we must change it into a series of units with a constant change of direction. The smaller we make these units the more accurate becomes our approximation; only if they could be made zero, would they be correct. But since we needs must conceive them as being ultimately concrete, rectilinear lines, they are treated as infinitesimals. The very idea is an unrealizable fiction, but it serves the purpose of a best possible approximation in describing a continuum in terms of discrete units.

But if the infinitesimal is unreal, because it is a fiction, how can it be useful? We must consider that it is a fiction which serves a purpose. There is a difference between "fiction" and "a fiction." Every mathematical concept is "a fiction" in the sense that it is not a thing, not an actual reality, not a concrete bodily object, but a product of thought, *ein Gedankenwesen,* as Kant calls it. If we treat a product of pure thought as if it were a concrete thing of objective reality we become involved into contradictions and are nonplussed. Here the indefiniteness of language proves a valuable help to mystagogues. We can make paradoxical statements about any mathematical term by an ambiguous use of such words as real, actual, true, etc. We may mean by "real" the concrete materiality of a thing, its definite efficiency in existence, or its objective significance. Thus the polar axis around which the earth turns may be called real or absolutely unreal, purely ideal or definite and actual.

In consideration of the paramount significance of relations (i. e. the purely formal aspect of things) the ancient mystic thinker of China said: "Existence makes things actual,[5] but the non-existent in them makes them useful" (*Lao-Tse's Tao Teh King,* Ch. 11).

One of Euclid's postulates declares that "the whole is greater than any of its parts," and we accept this truth for magnitudes; so far as I can see it can have no meaning when applied to items in which the quality of magnitude is absent. Take for instance the purely formal laws of the universe. They are a part of objective reality and yet their sphere of application may truly be said to be larger than that of the whole of which they form a part. By an *a priori* construction they have been developed in the subjectivity of the human mind and their sphere of efficiency applies to any possible world.

The same idea can be stated in religious terms thus: God is part of the All, yet God is greater than the All.

Professor Russell proposes for refutation a maxim shaped in imitation of this same postulate of Euclid. He says when speaking of the evasive nature of obviousness and self-evidence (p. 86):

> "For instance, nothing is plainer than that a whole always has more terms than a part, or that a number is increased by adding one to it. But these propositions are now known to be usually false. Most numbers are infinite, and if a number is finite you may add ones to it as long as you like without disturbing it in the least."

Mark the difference. "The whole is greater than any of its parts" and "the whole has always more terms than a part." Can we not describe the same thing in one term and in an infinite series of terms, as for instance:

$$1 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \textit{ad infinitum.}$$

[5] The common translation of *li* here is "profitable," and the etymology of the character which contains the roots "knife" and "harmony" indicates a meaning such as "cutting" or "efficient." The word is now used in the sense of "sharp." We might translate "pragmatic." But in the present passage it stands in contrast to "useful," and so I prefer the reading "actual" or "real."

The number of terms in which we cast our formula is not identical with the thing described. "One" is not infinite even though we can express it in an infinite series. If we ignore the difference between the thing and the terms in which it is expressed and count the terms numerically or quantitatively with an absolute disregard of their qualitative value, we are compelled to accept Zeno's solution of the problem, that Achilles can not overtake the tortoise in a running match. Professor Russell recapitulates this old conundrum thus (pp. 95-96):

"Let Achilles and the tortoise start along a road at the same time, the tortoise (as is only fair) being allowed a handicap. Let Achilles go twice as fast as the tortoise, or ten times or a hundred times as fast. Then he will never reach the tortoise. For at every moment the tortoise is somewhere, and Achilles is somewhere; and neither is ever twice in the same place while the race is going on. Thus the tortoise goes to just as many places as Achilles does, because each is in one place at one moment, and in another at any other moment. But if Achilles were to catch up with the tortoise, the places where the tortoise would have been, would be only part of the places where Achilles would have been. Here, we must suppose, Zeno appealed to the maxim that the whole has more terms than the part. Thus if Achilles were to overtake the tortoise, he would have been in more places than the tortoise; but we saw that he must, in any period, be in exactly as many places as the tortoise. Hence we infer that he can never catch the tortoise. This argument is strictly correct, if we allow the axiom that the whole has more terms than the part. As the conclusion is absurd, the axiom must be rejected, and then all goes well. But there is no good word to be said for the philosophers of the past two thousand years and more, who have all allowed the axiom and denied the conclusion."

While Professor Russell speaks of Zeno's conclusion as "absurd," and therefore rejects it, he regards the paradox of Tristram Shandy as a mere "oddity" which is a "paradoxical but perfectly true proposition." He says (pp. 96-97):

"The retention of this axiom leads to absolute contradictions, while its rejection leads only to oddities. Some of these oddities, it must be confessed, are very odd. One of them, which I call the paradox of Tristram Shandy, is the converse of the Achilles, and shows that the tortoise, if you give him time, will go just as far as Achilles. Tristram Shandy, as we know, employed two years in chronicling the first two days of his life, and lamented that, at this rate, material would accumulate faster than he could deal with it, so that, as years went by, he would be farther and farther from the end of his history. Now I maintain that, if he had lived forever, and had not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten. For consider: the hundredth day will be described in the hundredth year, the thousandth in the thousandth year, and so on. Whatever day we may choose as so far on that he cannot hope to reach it, that day will be described in the corresponding year. Thus any day that may be mentioned will be written up sooner or later, and therefore no part of the biography will remain permanently unwritten. This paradoxical but perfectly true proposition depends upon the fact that the number of days in all time is no greater than the number of years."

I hesitate to say that these two series are equal:

$1+1+1+1+1$ etc., without end, and

$365+365+365+365+365$ etc., also without end.

Yet if an infinite number of days will cover an infinite number of years the two series ought to be equal. I am afraid we shall all be hurled into infinity before we can find out the truth as to whether an infinity of days is as large as an infinity of years. If they are equal I should like to know what part the difference will play; since it will be the sum of an infinite series of 364 in each term.

I doubt whether the perverted form of Euclid's axiom is to be blamed (as Professor Russell thinks) for the deadlock to which Zeno's fallacy leads. I would say in explanation of the paradox that an infinite series need not be an actual infinitude. An infinite series is a mental operation, while an infinitude is the objective extension without end. An infinite series sometimes describes a very finite magnitude. For instance, $0.333\ldots\ldots$ is an infinite decimal fraction, but the infinite series of its terms (0.3+0.03+0.003....) does not involve that it represents an infinitude. It would take an infinitude to write all the decimals out in their completeness, but for that reason its value (say ⅓ of a second) is quickly passed and is not equal to any other infinite series, as for instance a third of an hour.

An infinite series is a function, and the essential feature of a function is the arrangement and not the number of its terms. If their number is limited we can sum up the *facit*; if it is unlimited or infinite we can never finish the function,—we can only approximate it or must let it stand, but the *facit* has nothing to do with it. It follows from this that two infinite series are not necessarily alike, because they are both infinite. They differ according to their terms and the arrangement of their terms. Here I am in full agreement with Professor Russell when he says, "It must not be supposed that all infinite numbers are equal" (p. 95), and rightly insists on the significance of "the way in which the terms are arranged" (p. 94), and the "particular type of order" (p. 97).

Professor Russell seeks the root of the trouble in the infinitesimal, but it lies there only if we forget the vague character of the infinitesimal, and expect it to be a definite magnitude to boot.

If we had to regard infinitesimals as actual and objective existences, there would be no such things as the next moment, and the smallest part. But in order to prove it we must be careful *not* to think of "moment" as a short yet definite measure of time. We could not prove our case if we said there is no such a thing as the next hour, or minute, or second, or jiffy. We must identify (as does Professor Russell) the word "moment" with the term infinitesimal, viz., the smallest possible fraction of time.

The same is true if we divide a piece of matter. We may come down to very small bits but shall never reach an

infinitesimal. "Nevertheless," says Professor Russell, "there *are* points, only they are not reached by successive division" (p. 91).

True, very true! Yet while in my opinion the propositions that "there is no next moment" and "there are no smallest particles" are due to the notion of the infinitesimal if conceived as an actual existence, Professor Russell attributes these very paradoxes to the abolition of the infinitesimal. He says (pp. 90-91):

"But at last Weierstrass discovered that the infinitesimal was not needed at all, and that everything could be accomplished without it. Thus there was no longer any need to suppose that there was such a thing....

"The banishment of the infinitesimal has all sorts of odd consequences, to which one has to become gradually accustomed. For example, there is no such thing as the next moment....

"The same sort of thing happens in space.... we never reach the infinitesimal in this way."

Professor Russell rejects the infinitesimal but accepts the infinite and he defines it, too. He says (pp. 92-93):

"The philosophy of the infinitesimal, as we have just seen, is mainly negative. People used to believe in it, and now they have found out their mistake. The philosophy of the infinite, on the other hand, is wholly positive. It was formerly supposed that infinite numbers, and the mathematical infinite generally, were self-contradictory. But as it was obvious that there were infinities—for example, the number of numbers—the contradictions of infinity seemed unavoidable....

"Twenty years ago, roughly speaking, Dedekind and Cantor asked this question [What is infinity?], and, what is more remarkable, they answered it. They found, that is to say, a perfectly precise definition of an infinite number or an infinite collection of things. This was the first and perhaps the greatest step. It then remained to examine the supposed contradictions in this notion. Here Cantor proceeded in the only proper way. He took pairs of contradictory propositions, in which both sides of the contradiction would be usually regarded as demonstrable, and he strictly examined the supposed proofs. He found that all proofs adverse to infinity involved a certain principle, at first sight obviously true, but destructive, in its consequences, of almost all mathematics. The proofs favorable to infinity, on the other hand, involved no principle that had evil consequences. It thus appeared that common sense had allowed itself to be taken in by a specious maxim, and that, when once this maxim was rejected, all went well.

"The maxim in question is, that if one collection is part of another, the one which is a part has fewer terms than the one of which it is a part. This maxim is true of finite numbers. For example, Englishmen are only some among Europeans, and there are fewer Englishmen than Europeans. But when we come to infinite numbers, this is no longer true. This breakdown of the maxim gives us the precise definition of infinity. A collection of terms is infinite when it contains as parts other collections which have just as many terms as it has. If you can take away some of the terms of a collection, without diminishing the number of terms, then there are an infinite number of terms in the collection."

I am somehow not satisfied with this definition; nor am I more enlightened through the example adduced for the sake of explanation (p. 93):

"For example, there are just as many even numbers as there are numbers altogether, since every number can be doubled.' This may be seen by putting odd and even numbers together in one row, and even numbers alone in a row below:

1, 2, 3, 4, 5, *ad infinitum.*
2, 4, 6, 8, 10, *ad infinitum.*

There are obviously just as many numbers in the row below as in the row above, because there is one below for each one above. This property, which was formerly thought to be a contradiction, is now transformed into a harmless definition of infinity, and shows, in the above case, that the number of finite numbers is infinite."

These several views of Professor Russell on the infinitesimal and the infinite do not seem to me quite consistent. But we shall hear from him again. He claims that there is a greatest infinite number while Cantor has offered a proof that there is none. Professor Russell says (p. 95):

"There is a greatest of all infinite numbers, which is the number of things altogether, of every sort and kind. It is obvious that there cannot be a greater number than this, because, if everything has been taken, there is nothing left to add. Cantor has a proof that there is no greatest number, and if this proof were valid, the contradictions of infinity would reappear in a sublimated form. But in this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work."

I believe that most mathematicians will side with Cantor. We claim that "the number of things altogether of every sort and kind," is not and can never be "the greatest of all infinite numbers." For suppose we would count all things of every sort and kind, and we had accomplished the task, we could add to it one or two or a few thousand units, we could multiply it with itself and so *ad infinitum.*

So far as I understand the nature of number there can no more be a highest number than there can be an end to space and time.

PROFESSOR BERTRAND RUSSELL'S CRITICISM.

Professor Russell's love of paradox renders his article interesting, but while it makes the reading of it pleasant, I am aware that it sometimes obscures the meaning. Having given it a careful and repeated perusal I am not sure that I have always rightly interpreted his humor. His censure of Euclid may be of this kind. We may agree better than it seemed to me at the first reading.

The problems concerning the foundations of geometry and of mathematics in general are by no means so definitely settled that one solution may be said to have acquired the consensus of the competent, and for this reason I feel that a little mutual charity is quite commendable. I have found it wanting mainly in those circles which represent the two extremes, the old-fashioned Euclideans and the new-fangled non-Euclideans; they scorn and condemn all who look at the problem through some other spectacles than their own. But I am glad to notice that Professor

Russell is not one of these. He can review considerately and kindly the work of one who differs from him on a subject to which he himself has given a great deal of attention. Therefore I here express publicly my recognition of the gentlemanly tone of Professor Russell's review, and having monopolized the floor myself in criticising him, I deem it but just to let him have his turn.

In the *Mathematical Gazette*, Vol. V, No. 80 (June-July, 1909), pp. 103-104, Professor Russell, speaking of my recent work, *The Foundations of Mathematics*, says:

"This book is a more or less popular exposition of a philosophy of geometry which is, in its main outlines, derived from Kant. The main title, if uncorrected by the sub-title, would be somewhat misleading, since the foundations of arithmetic and analysis are not discussed, but only the foundations of geometry. The author begins by a brief account of the development of non-Euclidean geometry, which is followed by much longer chapters "on the philosophical basis of mathematics" and on "mathematics and metageometry." The historical chapter, though it does not profess to give more than a sketch, might with advantage have been enlarged by some account of projective geometry and the projective treatment of metrics. Dr. Carus speaks always as though non-Euclidean straight lines were not really straight, but were merely called straight out of wilfulness. The projective treatment shows, better than the metrical, wherein the straight lines of non-Euclidean spaces agree with those of Euclid, and ought therefore not to be omitted even in a mere outline. It would seem also that Dr. Carus regards a three-dimensional non-Euclidean space as necessarily contained in a four-dimensional Euclidean space, for he asks "what Riemann would call that something which lies outside of his spherical space," apparently not realizing that spherical space does not require anything outside it.

"The author's philosophical theory of geometry may be summarized as follows. Geometry, like logic and arithmetic, is *a priori* but it is not *a priori* in the same degree as logic and arithmetic. There is the *a priori* of being and the *a priori* of doing, and geometry belongs to the latter: it is derived from the contemplation of motion, and can be constructed from the 'principles of reasoning and the privilege of moving about.' We know *a priori* what are the possibilities of motion; thus, although there is nothing *logically* impossible about the assumption of four dimensions, yet 'as soon as we make an *a priori* construction of the scope of our mobility, we find out the incompatibility of the whole scheme.' The *a priori* is identical with the purely formal, which originates in our minds by abstraction; it is applicable to the objective world because the materials of formal thought are abstracted from the objective world.

"Most of the arguments in the book lead one to expect that Euclid will be declared to be certainly alone valid as against non-Euclidean geometry, yet this is not the conclusion drawn by the author. He says: 'The result of our investigation is quite conservative. It re-establishes the apriority of mathematical space, yet in doing so it justifies the method of metaphysicians in their constructions of the several non-Euclidean systems.... The question is not, "Is real space that of Euclid or of Riemann, of Lobatchevsky or Bolyai?" for real space is simply the juxtaposition of things, while our geometries are ideal schemes, mental constructions of models for space measurement. The real question is, "Which system is the most convenient to determine the juxtaposition of things?"' (p. 121). Yet a few pages later he says: 'The theorem of parallels is only a side issue of the implications of the straight line' (p. 129). It is not clear how these statements are reconciled, for the earlier statement seems to imply that there is no 'theorem' of parallels at all.

"A few of the author's assertions are somewhat misleading. For example, he states, as a fact not open to controversy, that Euclid's axiom or postulate of parallels originally occurred first in the proof of the twenty-ninth proposition, not being mentioned either among the axioms or among the postulates (p. 2). On the other hand, Stäckel and Engel (*Theorie der Parallellinien*, p. 4) say that, following Heiberg, they do not regard the postulate of parallels as a later addition, which would seem to show that Dr. Carus's opinion is at least open to question. Again he says (p. 84): 'While in spherical space several shortest lines are possible, in pseudo-spherical space we can draw one shortest line only.' As regards spherical space, the more exact statement is that in general only one shortest line can be drawn between two given points, but when the two points are antipodes, an infinite number of shortest lines can be drawn between them.

"The book concludes with an epilogue, in which the existence and attributes of the Deity are deduced from the nature of mathematical truth."

With reference to Professor Russell's several comments I will make these statements:

(1) The title of my book read originally "The Foundation of Geometry," but since this designation had been forestalled by Professor Hilbert's book I changed it to "The Foundations of Mathematics" with the subtitle "A Contribution to the Philosophy of Geometry," to make up by it for what may be misleading in the main title.

(2) Though I will grant that a discussion of projective geometry might be added to advantage in an exposition of non-Euclidean geometry, I doubt whether it will help us much in laying the foundation of geometry. I am inclined to think that it might complicate the problem and confound the issue.

(3) I am indeed of the opinion that the use of the term "straight line" had better be limited to the straight line of Euclidean space and that its analogies in other spaces should be named "straightest lines" or be designated by any other term that might be deemed appropriate.

(4) I conceive every kind of space conception as independent and grant that none of them ought to be thought of as being constructed in Euclidean space. But if space is a scope of motion, I can not think of a space that is limited. Spherical space ought to be conceived as possessed of a spherical drift, but for that it ought to be infinite. If it is not infinite, I would ask the question, what is outside? In my opinion we can not get rid of infinitude. The straightest lines in spherical space would not be infinite. They would be merely boundless. Outside of every boundless spherical line we must be able to construct other lines or spherical surfaces and thus spherical space would be as infinite as Euclidean space. I may be wrong but I am willing to learn.

(5) The passage on page 84 is an obvious mistake. When I wrote it I had in mind the Mercator projection of the globe where both the meridians and the parallels assume the same straightness as the straight lines of Euclidean space. The parallels (so called by geographers) are not shortest lines, but if the parallels on the globe, because they represent straight lines in the Mercator projection, were called straightest lines, we could make them enclose a space with shortest lines on the sphere, or with the parallels of another equatorial system. Of course not being truly shortest lines, the statement is a mistake and I am much obliged to Professor Russell for having called my attention to it.

PARALLELISM AND INFINITY. A COMMENT ON MR. FRANCIS C. RUSSELL'S THEOREM.

Mr. Francis C. Russell, an American namesake of Professor Bertrand Russell, of Cambridge, England, stands up so doughtily for Euclid that he has excited the wrath of non-Euclideans. In his retort courteous to the strictures of his critics he proposes a theorem which would upset Lobatchevsky and Bolyai, if the right angle were and remained the same in non-Euclidean space as in plane geometry. But this is exactly the crux. We need more rigidity in the use of terms.

Our geometricians, Euclid as well as the non-Euclideans, have not always defined with sufficient precision all the names and notions which they introduce. A right angle according to my conception of geometrical notions belongs to the important class of boundary conceptions which on account of their uniqueness become standards of measurement.

The straight line (corresponding to a crease in a sheet of paper folded upon itself) halves the plane, while the right angle (corresponding to the folded sheet of paper again folded upon itself) represents the halved half of the plane. There are innumerable curves, but only one straight line; and there are innumerable acute and innumerable obtuse angles, but only one right angle. Accordingly we may define the right angle as the angle which represents one-quarter of the entire sweep of direction round a common center. The right angle on a sphere is smaller than in plane geometry; it increases and decreases with the length of the radius and becomes approximately equal to the plane right angle only when the radius becomes infinitely great.

Mr. Russell proves in his theorem that the right angle remains a right angle, but the Russell theorem holds good only for Euclidean space. The right angles in other spaces follow the law of their spaces.

In the same way the term parallel has another sense in plane than in projective geometry. We are told that "lines which meet at infinity are called parallel," but if two lines are truly parallel will they not remain parallel even in infinity? We may freely grant that they will meet at infinity, but it would be better not to introduce this feature in the definition.

The cause of much trouble must be sought in the use of Euclidean terms in a non-Euclidean sense. By a straight line mathematicians formerly understood the straight line in the Euclidean plane, and if we now become acquainted with other lines called straight because they somehow correspond to the straight line of the Euclidean plane, the layman who is only superficially acquainted with the new geometry is naturally puzzled. It is therefore advisable to call the non-Euclidean straight line a "straightest line," or to give it some other suitable name so as to distinguish it from the straight line in the plane.

The same is true of parallels. So far as my linguistic feeling is concerned I cannot overcome the original meaning of the etymology of parallel. Parallel lines are to me lines which remain the same distance apart; thus rail-

road tracks, whether straight or curved, are parallel; and since this running side by side is the original sense of the word, they deserve to be so called. The geographer too uses the word parallel in its etymological meaning. If straight lines are parallel (i. e., keep at the same distance) they do not meet even in infinity,—although I will grant anything for infinity.

Infinity is the land of mathematical hocus pocus. There Zero the magician is king. When Zero divides any number he changes it without regard to its magnitude into the infinitely small; and inversely, when divided by any number he begets the infinitely great. In this domain the circumference of the circle becomes a straight line, and then the circle can be squared. Here all ranks are abolished, for Zero reduces everything to the same level one way or another. Happy is the kingdom where Zero rules!

I do not say that the notion of infinity should be banished; I only call attention to its exceptional nature, and this so far as I can see, is due to the part which zero plays in it, and we must never forget that like the irrational it represents a function which possesses a definite character but can never be executed to the finish. If we bear in mind the imaginary nature of these functions, their oddities will not disturb us, but if we misunderstand their origin and significance we are confronted by impossibilities.

[illegible] in infinity all is reduced to a democratic sameness [illegible] points on two parallel lines coincide when they lie at infinity. And the same is true of planes. Suppose we lay down a plane on the absolutely smooth surface of the ocean. It would be tangential to the earth, and the entire range of points at infinity would lie on a line which we will call its infinity-horizon. It is the circumference of a circle, all the diameters of which are infinite straight lines. Now consider that just as two parallel lines cut each other at infinity, so two parallel planes cut each other in a line at infinity, and we must accept the conclusion that all the infinite number of parallel planes of the entire system of all possible parallel planes upward to an infinite distance and downward to an infinite distance meet at an infinite distance in one line. All their infinity-horizons coincide. Another such a system of planes set at right angles on the horizontal level would behave in the same way. All the perpendicular planes meet in a line which we may call their infinity- meridian. The infinity-meridian is a line which at infinity passes through all the horizontal planes at the point where it cuts their infinity-horizon.

Here the point at infinity is a line and the line is a point.

The infinity-meridian of all the infinite number of perpendicular planes cuts the infinity-horizon of every single plane of the infinite number of horizontal planes, and there are no points in the infinity-meridian which do not cut some of the horizontal planes at infinity. But since all the horizontal planes have only one infinity-horizon, the range of points of the infinity-meridian all lie in the very same point of the infinity-horizon. In other words, the infinity-meridian is one point on the infinity-horizon, and with this point the entire range of its other points will be found to coincide. Thus this line (the infinity-meridian) shrinks into one point and nothing of the infinite extent of the line lies outside this point. Inversely the infinity-horizon is a mere point on the infinity-meridian.

The mathematician may well turn mystic when he moves in infinitudes.

KANTISM AND SPACE.

In a recent number of *The Mathematics Teacher*, Prof. E. D. Roe, Jr., publishes an article under the title "Some Thoughts on Space." He is a Kantian who maintains that "it does not seem that Kant's fundamental principle

that time and space in general are necessary forms of intuition is overthrown." He says (p. 35):

"The simplest hypothesis is not that a handing over to a blank, which is unintelligible, took place, but that reason by its own spontaneity, at least, acted according to the laws of its own activity, and applied its forms and categories to the matter of experience..... If the mind is not constituted so to act space will not result no matter how much experience is had. A cow has the same empirical occasions, but does any one imagine that mathematical space is in a cow's mind, or ever will be, or that it could be revealed to or handed over to the cow, or by any instruction be conveyed to or gotten into the cow's consciousness? If experience could cause its genesis it would be there, for a cow has as much experience as we, doubtless more, because the cow has nothing but experience and all of it in time and space."

Professor Roe differs from Kant in one point. He says (p. 36):

"Kant denied their objective reality. But he should not have done so as by his theory he did not know what was external to the mind. He should have neither denied nor asserted this."

I will not enter here into a discussion of either Kant's or Mr. Roe's doctrine of space. I will only quote what Mr. Roe says of my position:

"The preceding conclusion was reached independently. My attention has since been called to the book of Dr. Carus, *The Foundations of Geometry*, Chicago, The Open Court Publishing Co., 1908. It is a great pleasure to recognize in him a friend and not an antagonist. The results here reached seem to agree with those of Dr. Carus, though the method and standpoint are a little different."

What Kant calls "the empirical occasions" are the single and innumerable experiences we have in life. They are the facts from which we derive our general idea of motion. This general idea of motion is an abstraction. It is not real motion, but the thought of motion. All consideration of energy is omitted from it, it is what Kant would call "pure motion," and the *a priori* constructions of it, such as lines, angles, triangles, etc., have nothing to do directly with any empirical occasions. The cow can not think in abstractions, and this is the reason why a cow knows nothing of mathematical space.

Mr. Roe objects to the term "abstraction," because it "conveys the idea of unreality." First I would answer that there is no harm in this, for the whole of mathematics is an ideal construction, and it prospers well in the atmosphere of unreality—unreality in the sense that it does not consist of concrete material objects. But secondly I would add that abstractions describe features of real things, and though abstractions as such are not concrete they represent qualities which are real.

Mr. Roe introduces the word "spontaneity" to serve in the place of abstraction, but we are little helped thereby. Spontaneity means self-motion, and it is to be feared that it will involve us into questions as to the nature of mental activity. It is certainly not a simpler conception than pure motion; and actual motion of some kind is absolutely needed in order to give us the abstract notion of either motion or spontaneity. Professor Roe suggests:

> "Why might not one lie perfectly still with eyes closed and receive tactile sensations on different parts of his body and some notion of here and there be called out without the necessity of subjective motion?"

If the notions of "here and there" could originate under these conditions they would be the product of referring to tactile impressions "here" and other impressions, "there." A normal man would use his hand to localize sensations. We do not know which tooth hurts us until we touch the sore spot, but suppose the localization were roughly done internally, we could accomplish it only by allowing our attention to move about from the place of one sensation to that of another. Motion is indispensable for any space-construction.

I will abstain from discussing other points and will only say that such words as "faculty," which Mr. Roe introduces, had better be avoided.

* * *

Professor Bertrand Russell makes the statement that I have derived my philosophy of geometry in its main outlines from Kant, but that is true in a certain sense only. I have started from Kant and retain much of his terminology, but in the most essential points I have come to conclusions diametrically opposed to his.[6] I might as well consider myself a disciple of Grassmann, although I did not become acquainted with his extension theory until later in life; but he was my teacher in mathematics, and I may unconsciously have imbibed from him many notions which like fertile seeds grew up in my mind and are now inextricably intertwined with my own thoughts.

The statement is often made that our mathematical conceptions depend upon our senses. If we had different eyes or other organs of sense, it is claimed, we would have a different notion concerning space. But this is true only so far as our physiological space-conception is concerned, and even there the modification would be slight. It is difficult to prognosticate what space-notions we would have if we were endowed with an electric sense, but it stands to reason that even the perception of electric shocks or the ability to perceive a discharge of electric shocks upon our surroundings would change nothing in our notion of space, for it would have to be interpreted ultimately with the assistance of the sense of touch, which is and will remain the foundation of all sense-perception. I am convinced that the ability to move with great rapidity, which would be acquired by the faculty of flight, would modify our space-conception more than the possession of electric or

[6] For details see my discussion of Kant's doctrines in my little book *Kant's Prolegomena*.

any other additional sense-organs. After all, our notion of space is ultimately based on the self-observation of our own motion. Without motion no space-conception.

Physiological space is dependent to a great extent on our physiological constitution, and the latter again depends upon the conditions in which we live. The feelings "upward" and "downward" are decidedly different, being subject to the factor of gravitation; and mathematically congruent figures in different positions appear different to us on account of the distinction we make between upward and downward, high and low, right and left.[6]

Mathematical space differs from physiological space in being of a more abstract nature. For its construction we need the idea of pure motion alone, which is treated according to the rules of consistency, analogous to pure logic. Accordingly pure mathematics does not depend upon the senses but is the product of the mind. If rational beings, differing from ourselves, have developed on other planets, they might have different notions of physiological space than we have, but they would have the same logic, the same arithmetic, the same geometry, and all the complications derived therefrom.

Helmholtz once pointed out that rational beings who were moving on a sphere would develop a spherical geometry; but, strange to say, he at the same time overlooked the fact that man is actually living on a sphere, and yet our geometry, as it developed in history, is plane and not spherical. We shall scarcely be mistaken when we say that were we to visit other planets we should nowhere find a race of rational beings who developed either the geometry of Lobatchevsky, or Bolyai, or any other non-Euclidean system, before they would discover that plane geometry was also possible. The non-Euclidean geometries prove

[6] For details see Mach's investigations on "Physiological as Distinguished from Geometrical Space," in his excellent little book *Space and Geometry* (Chicago: Open Court Pub. Co., 1906).

that mathematical concepts of any kind are *a priori* in Kant's sense, which means that they are purely mental constructions. The invention of non-Euclidean systems is not useless, for they enable us to generalize our space-conception and subsume geometrical propositions under larger conceptions of different or higher manifoldnesses. We are not able to visualize some of the non-Euclidean spaces, which means we cannot form definite sense-perceptions of them; but we can think them and establish their several laws in abstract formulas. This generalization is a gain because it enables us better to understand the nature of mathematics in general as well as in its particular propositions. However, metageometricians go too far and misunderstand the significance of non-Euclidean geometries, if they treat mathematical space-conceptions as actualities and expect the rival claims of Euclidean and non-Euclidean systems to be decided before the tribunal of the *a posteriori*, i. e., of experience.

CONCLUSION.

Having laid the foundation of geometry without resorting to axioms, merely through the function of pure motion, the latter being ultimately derived from experience through abstraction, by omitting everything particular, I feel confident that I have furnished a conception which satisfies all demands and will be serviceable for all practical purposes. I avoid the mysticism which necessarily results from other interpretations. I may have overlooked applications of importance but I feel confident that all difficulties can be overcome, and that in the main my solution is on the right track.

Dixi et salvavi animam meam.

EDITOR.