

Digital Currency Runs

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Abstract

This paper studies the potential impact of digital currency on the banking system. Banks are resilient to aggregate asset and liquidity risk in an economy with digital currency, as with fiat money, but are at risk of interbank market lending freezes exacerbated by price and debt deflation spirals. Rather than displacing banks, digital currency threatens a new form of banking crises caused by disintermediation runs. A central bank can act as lender of last resort to prevent these crises in an economy with central bank issued digital currency, as with fiat money, but not with private issued digital currency.

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1. Introduction

The rapid development of digital currency has prompted widely acclaimed interest in its potential impact on the financial system and the economy.¹ In response, central banks worldwide are considering issuing their own digital currency.² Questions have emerged about whether digital currency may eventually displace the banking system, and concerns have arisen about whether digital currency may create fragility in the financial system.³

This paper develops a model of banking under a modern fiat monetary system and considers how the banking system would compare under an alternative monetary system based on digital currency. I show that a fiat monetary system allows for banking stability against aggregate asset and liquidity risk but is subject to crises from interbank market lending freezes. An economy based on a digital currency would feature a similar fractional reserve banking system. A key distinction of an economy based on digital currency is not the displacement of banking, but rather the new risk of disintermediation in the form of withdrawal runs based on the ability of depositors to efficiently withdraw and hold digital currency outside the banking system. A key difference also arises according to whether the digital currency is issued by a central bank, which I refer to as a public digital currency, or is not government based, such as bitcoin, which I refer to as a private digital currency. A central bank can act as lender of last resort to prevent financial crises caused by interbank freezes and withdrawal runs in an economy tied to a public digital currency, as with traditional fiat money, but not in an economy tied to a private digital currency.

An economy with a monetary system that is based on a private digital currency instead of a central bank fiat money is a viable possibility, as suggested by Raskin and Yermack (2016). As they explain, the advent by Nakamoto (2008) of bitcoin was designed to be supplied at a determined rate of growth to disallow discretion in the money supply accorded to central banks. Meanwhile, central banks may begin issuing their own sovereign digital currencies.⁴ Raskin and Yermack (2016) also point out expectations

¹Early references include Raskin (2012, 2013), White (2014), Kim (2015).

²See Bech and Rodney (2017), Bordo and Levin (2017), Broadbent (2016), Cawrey (2014), Jack and Suri (2011), Popper (2015), and Rosenfeld (2015).

³For example, see Winkler (2015) and Nelson (2017).

⁴See Bech and Rodney (2017), Bordo and Levin (2017), Broadbent (2016), Cawrey (2014), Jack and Suri (2011), Popper (2015), and Rosenfeld (2015).

that digital currency would disintermediate banks by ending fractional reserve banking. The establishment of either a public or private digital currency in an economy could allow for holding money and an efficient electronic means of payment without use of the banking system.

This paper has three goals. The first is to show novel results about the resiliency and fragility of the banking system based on an elastic price level that arises in a parsimonious model of the economy with fiat money, which allows for a nominal unit of account, in contrast to a real economy. The second is to show a primary distinction of the banking system operating with a digital currency, whether private or public, rather than fiat money, which is that it allows for disintermediation-based withdrawal runs. The third is to show a primary distinction between a private and public digital currency, which is that a central bank is able to prevent banking crises by acting as lender of last resort under a regime with a public digital currency, as with fiat money, but not with a private digital currency regime.

First, I show that in an economy with an elastic price level, arising from a nominal unit of account based on fiat money in a general equilibrium setting, the banking system is not subject to traditional bank runs driven by panics of depositor runs, such as occur in the seminal theory of bank runs in Diamond and Dybvig (1983). I show the novel result that such panic based runs do not occur even with aggregate asset and liquidity risk. As unit of account, money allows for elastic prices and a flexible real value of nominal bank deposits that provides risk sharing in the economy. Banks lend to firms, which optimize over the provision of real goods and asset liquidation based on equilibrium prices in the goods market. This liquidity and flexibility within the financial system also helps to avoid financial distress caused by liquidity-based runs on the banking system and aggregate risk that can cause insolvency of the banking system.

Instead, the banking system is fragile because of idiosyncratic bank liquidity shocks that allow for the potential of an interbank market lending freeze, which acts essentially as a panic among banks with each other. In an interbank freeze, illiquid banks cannot borrow on the interbank market and have to liquidate. The liquidation of illiquid banks lead to a decrease in liquidity for all banks and can lead to systemic contagion runs on the entire banking system. While price elasticity in the economy makes the financial system resilient to aggregate liquidity and solvency shocks, this increased price elasticity implies that when banks are liquidated, there is greater price deflation in the economy. Price deflation creates debt deflation, in which banks face a higher real cost of their deposits,

firms have a higher real debt burden that increases liquidation, and a downward spiral can lead to systemic contagion runs on the entire banking system.

Second, I show that in contrast to suggestions that a private or public digital currency would supplant banking, banking still occurs because of the benefits of maturity and risk transformation of illiquid assets for the efficient provision of liquidity to the economy. Private digital currency is a form of outside money that banks can hold as reserves, similar to the case of fiat money as reserves, to enable standard fractional reserve banking.

Third, I show that rather than digital currency disintermediating banking from the outset, digital currency enables a distinct form of panic runs on the banking system from the ability of depositors to withdraw and hold currency outside of the banking system. Since digital currency provides an outside form of money that can be withdrawn, stored, and efficiently used as a direct means of payment within the economy, withdrawal runs can deplete digital currency from the banking system and cause a complete banking collapse. With either withdrawal runs or interbank market freezes, there are failures of illiquid but otherwise solvent banks. In an economy with a publicly issued digital currency, as with fiat money, a central bank can provide an elastic supply that allows it to avert financial crises by acting as lender of last resort to illiquid but otherwise solvent banks. When such banks do not have to liquidate, the downward price and debt deflation spiral does not occur. Whereas, in an economy with a privately issued digital currency, the central bank is not able to create fiat money and loses its ability to act as lender of last resort, which allows interbank freezes and withdrawal runs to occur.

Indeed, as Raskin and Yermack (2016) point out, the Federal Reserve was originally created for the primary purpose of being able to provide an “elastic supply of currency” in order to act as a lender of last resort. But, the Fed’s discretion to increase the money supply in response to financial and economic distress has often come under pressure since the founding of the Fed. The earliest call for a privately issued digital currency to constrain the elastic supply of money is likely by Milton Friedman. As early as 1999, Friedman famously appeared to foresee and welcome the opportunities for a digital currency to be supplied inelastically in an algorithmic manner according to an automated rule in order to constrain monetary policy discretion, as described by Raskin and Yermack (2016).

Digital currency has been recently studied, along with blockchain technology utilized with distributed decentralized ledgers more broadly, in the rapidly growing finance and

economics literature on fintech. The viability of digital currency as a stable form of money depends importantly on the ability for an underlying blockchain technology to support consensus for transactions and the cost of a blockchain technology for transactions.

More generally, the potential for an even robust private digital currency to be adopted as a meaningful form of money is a highly debated question. For example, bitcoin and other private digital currencies are currently viewed by many as primarily a means for black markets, speculative and ill-defined investment, such as with initial coin offerings, or an asset bubble, rather than as a form of money. Nonetheless, several less developed countries have wavered between the extremes of officially supporting the adoption of private digital currency, such as bitcoin, for its potentially greater stability as a monetary form, and banning its use because of its possible ability to take the place of a country's own fiat money. Developed countries are also considering developing public digital currencies in part because of the perception for private digital currency to potentially displace fiat money. Whether private digital currency could displace central bank fiat money may be in large part a political question. In addition, the ability for private digital currency to be widely adopted as money may be viewed in part as an economic coordination problem. For example, bitcoin has displayed extreme price volatility, limited acceptance and use, and large competition from other private digital currencies, which in turn further limits its acceptance and use. However, if a private digital currency were to be broadly adopted in an economy as the predominant monetary system, such widespread acceptance and use may lead the private digital currency to have a stable value further supporting its use.

In this paper, I consider two specific features of digital currency that have particular relevance for the banking system. First, digital currency has the potential to be used as an efficient means of electronic payments without utilizing the banking system. For this reason, I make the simplifying assumption of no transactions costs for payments in the economy and financial system made by using either digital currency as outside money or bank deposits as inside money. Second, a private digital currency has the potential to be supplied in a determined quantity without discretion in the money supply that central banks hold. To further focus on the implications of these two features of digital currency for the banking system, I do not examine questions of competition between a private digital currency with a public digital currency and fiat money, or among different private digital currencies. Rather, I analyze the fragility of the banking system by comparing

the results of an economy under the establishment of either (i) a fiat money supply without digital currency, (ii) a public digital currency, or (iii) a private digital currency.

The academic literature on fintech is growing rapidly. Yermack (2014) provides an early introduction to much of the academic finance literature on bitcoin and blockchain technology. Raskin and Yermack (2016) argue that a central bank digital currency would enable households to hold deposits directly in such a public digital currency instead of commercial banks and that debates over these issues is demanding a resurgence in studying classical monetary economics.⁵ Recent studies include Aloosh (2017), who studies digital currency from an asset pricing perspective. Cong and He (2017) analyze the potential for blockchain technology to allow for more efficient financial contracting. Malinova and Park (2017) develop a model of trading in financial markets to show that distributed ledgers allow for managing the level of transparency in trading to increase investor welfare.

The theory of financial fragility begins with the seminal analysis of bank runs by Diamond and Dybvig (1983). Recent studies have examined dynamic models of bank runs;⁶ banking stability tied to efficient risk management (DeAngelo and Stulz, 2015); banking efficiency and fragility in relation to the information sensitivity of deposits (Dang et al. (2013), contracts relative to markets (Allen and Gale, 2004), and central bank interest rate policy (Freixas et al., 2011); and global games information signal structures (Goldstein and Pauzner, 2005).

Research that examines the potential efficiency and fragility of nominal bank contracts includes most recently Allen et al. (2014). They develop a model of banking with nominal contracts in which a central bank provides a provision of lending money intra-day that allows banks to provide optimal liquidity and the first best allocation in the face of idiosyncratic and aggregate liquidity shocks and aggregate asset return shocks, but the study does not examine the fragility of nominal contracts. Diamond and Rajan (2006) and Champ et al. (1996) examine nominal bank contracts to show that runs and contagion occur due to withdrawals of currency out of the banking system based on purchases of goods that must be made with currency. Diamond and Rajan (2006) further show that nominal contracts do not protect from bank runs caused by heteroge-

⁵See von Mises (1912) as well as recent classics such as Hayek (1976) and Mundell (1998).

⁶Brunnermeier and Sannikov (2016) study debt deflation in a financial intermediation theory of money. Other recent dynamics of bank runs inspired from the recent financial crisis include Gertler and Kiyotaki (2015), Martin et al. (2014), Brunnermeier and Oehmke, and He and Xiong (2012).

neous shocks in asset returns. Skeie (2004, 2008) studies the efficiency and fragility of nominal contracts based on a banking model in which a central bank intervenes intraday with a provision of money. A long line of the theory of bank runs also includes studies in the context of interbank lending;⁷ the role of lending money between banks, central bank lending and injections of money, and demand deposits paid in money in models of real bank deposits;⁸ liquidity runs and bank insolvency tied to bank lending contracts;⁹ systemic risk triggered from idiosyncratic bank losses;¹⁰ and interbank payments and lending operating through clearinghouse systems for transferring and settling payments between banks.¹¹

The paper proceeds as follows. The model of banking in a economy based on fiat money, or a public or private digital currency, and the results of optimal liquidity and banking resilience with aggregate asset and liquidity risk, are presented in Section 2. The potential for financial crises based on interbank market freezes and withdrawals runs, and the distinction between an elastic and inelastic digital currency that allows for the central bank to act as lender of last resort, are analyzed in Section 3. Section 4 provides concluding remarks. All proofs are contained in the appendix.

2. Model

2.1. Real economy

The model has three periods, $t \in T_{0,1,2} \equiv \{0, 1, 2\}$. There is a large number of ex-ante identical consumers who are endowed with g_0^C goods at $t = 0$. Goods g_t can be stored at $t \in T_{0,1} \equiv \{0, 1\}$ for safe, short-term liquidity with a return of one at $t + 1$. Goods can also be invested as risky, long-term illiquid assets a_0 at $t = 0$. An amount of these assets $a_1 \leq a_0$ can be liquidated at $t = 1$ for a salvage return of $r_1 \in (0, 1)$ at $t = 1$. The remaining assets $a_0 - a_1$ that are not liquidated have a random return $r_2 \in (0, r_2^{\max})$ at $t = 2$ with expected return $E[r_2] = \bar{r}_2 > 1$.

A random fraction $\lambda \in (0, 1)$ of consumers have a privately observed liquidity shock

⁷Bhattacharya and Gale (1987), Bhattacharya and Fulghieri (1994), Allen and Gale (2000a) and Diamond and Rajan (2005).

⁸Allen and Gale (1998), Gale and Vives (2002), Freixas et al. (2000, 2003), Freixas and Holthausen (2001), Rochet and Vives (2003), and Chang and Velasco (2000).

⁹Holmström and Tirole (1998) and Diamond and Rajan (2005).

¹⁰Rochet and Tirole (1996) and Aghion et al. (2000).

¹¹Furfine (1999), Henkel (1999), Flannery (1996) and Hancock and Wilcox (1996).

and need to consume at $t = 1$, where $E[\lambda] = \bar{\lambda}$. These early consumers have utility given by $U = u(c_1)$. The remaining fraction $1 - \lambda$ of consumers do not receive a liquidity shock. These late consumers have utility $U = u(c_1^i + c_2^i)$. The utility function $u(\cdot)$ is assumed to be twice continuously differentiable, strictly concave, satisfy Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$, and have a coefficient of relative risk aversion (CRRA) $\frac{-cu''(c)}{u'(c)} > 1$. The random joint aggregate liquidity shock and asset return shock state (λ, r_2) is realized and observable at $t = 1$ but not verifiable.

2.1.1. First best

The full-information, first best allocation follows from the optimization:

$$\max_{g_0, g_1, a_0, a_1} \quad EU = E[\lambda u(c_1) + (1 - \lambda) u(c_1^i + c_2^i)] \quad (2.1)$$

$$\text{s.t.} \quad g_0 \leq g_0^C - a_0 \quad (2.2)$$

$$\lambda c_1 \leq g_0 - g_1 + a_1 r_1 \quad (2.3)$$

$$(1 - \lambda)(c_1^i + c_2^i) \leq (a_0 - a_1)r_2 + g_1. \quad (2.4)$$

Proposition 1. *The optimal consumption for early and late consumers, c_1^* and c_2^* , respectively, requires storage at $t = 1$, $g_1^* > 0$, for a relatively low joint state $(\check{\lambda}, \check{r}_2)$, and liquidation, $a_1^* > 0$, for a relatively high joint state (λ, r_2) .*

The first-order conditions and binding constraints for the planner's optimization give optimal consumption according to

$$E[u'(c_1^*)] = E[r_2 u'(c_2^*)] \quad (2.5)$$

$$c_1 = c_1^* = \frac{g_0^* - g_1^* + a_1^* r_1}{\lambda} \quad (2.6)$$

$$c_2^i = c_2^* = \frac{(a_0^* - a_1^*)r_2 + g_1^*}{1 - \lambda} \quad (2.7)$$

$$c_1^i = 0. \quad (2.8)$$

The first line above gives the Euler equation showing that in expectation, the ratio of marginal utilities between $t = 1$ and $t = 2$ is equal to the marginal rate of transformation r_2 .

Optimal liquidity risk-sharing among consumers decreases consumption risk with a CRRA greater than one. Early consumers have an expected consumption $E[c_1^*]$ that

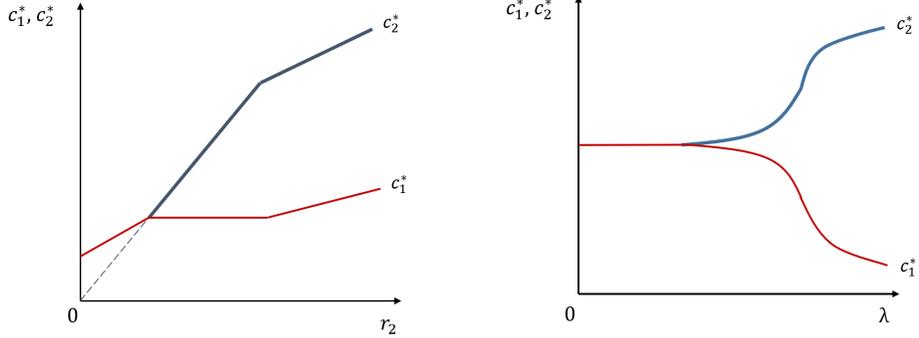
is greater than their endowment, g_0^C , implying an expected return on endowment for early consumers that is greater than the return on storage. This is implemented with an optimal quantity of $t = 0$ storage, g_0^* , that is greater than the endowment of the expected fraction of early consumers, $\bar{\lambda}g_0^C$. Whereas, if CRRA was less than one, optimal liquidity risk-sharing among consumers would be outweighed by the expected return on illiquid assets, \bar{r}_2 . Storage g_0 would be lower than $\bar{\lambda}g_0^C$ in order to provide for greater asset investment, a_0 . This would lead to an increase in consumption risk for consumers, with early consumers receiving an expected return on endowment that is less than one in order to provide late consumers with an expected return on endowment that is greater than the expected asset return \bar{r}_2 .

The optimal storage, liquidation, and consumption allocation depends on the joint realization of (λ, r_2) . Define consumption if there is no storage or liquidation for any realization of (λ, r_2) , $g_1(\lambda, r_2) = a_1(\lambda, r_2) = 0$, as $\tilde{c}_1 = \frac{g_0^*}{\lambda}$, $\tilde{c}_2 = \frac{a_0^* r_2}{1-\lambda}$. For $u'(\tilde{c}_1) < u'(\tilde{c}_2)$, there is positive storage $g_1^* = (1-\lambda)g_0^* - \lambda a_0^* r_2 > 0$ to equalize marginal utilities between early and late consumers such that $u'(c_1^*) = u'(c_2^*)$. As a result, $c_1^* = c_2^* = g_0^* + a_0^* r_2$. This outcome occurs for a low enough joint realization of (λ, r_2) , which can be expressed as $r_2 < \check{r}_2(\lambda) \equiv \frac{(1-\lambda)g_0^*}{\lambda a_0^*}$ and $\lambda < \check{\lambda}(r_2) \equiv \frac{g_0^*}{g_0^* + a_0^* r_2}$. When the illiquid asset return or the aggregate liquidity need for early consumers is small enough, positive storage of goods from $t = 1$ to $t = 2$ enables late consumers to share equally with early consumers in the total goods available at $t = 1$ and $t = 2$. The marginal rate of substitution between late and early consumers equals the marginal rate of transformation of one on storage between $t = 2$ and $t = 1$.

For $u'(\tilde{c}_1) > \frac{r_2}{r_1} u'(\tilde{c}_2)$, which holds with an implicit $(\hat{\lambda}, \hat{r}_2)$ for $r_2 > \hat{r}_2(\lambda)$ and $\lambda > \hat{\lambda}(r_2)$, such a high enough joint realization of (λ, r_2) implies there is instead positive liquidation $a_1^* > 0$ implicitly defined by $u'(c_1^*) = \frac{r_2}{r_1} u'(c_2^*)$. When the illiquid asset return or the aggregate liquidity for early consumers is large enough, asset liquidations allows for early consumers to share in part of the abundance of goods that are available at $t = 2$. The marginal rate of substitution between late and early consumers equals the marginal rate of transformation between assets' return at $t = 2$ and liquidation return at $t = 1$.

Otherwise, for $u'(\tilde{c}_1) \in \left[u'(\tilde{c}_2), \frac{r_2}{r_1} u'(\tilde{c}_2) \right]$, for moderate realizations of (λ, r_2) , there is no storage or liquidation, $g_1^* = a_1^* = 0$, hence $u'(c_1^*) \in \left[u'(c_2^*), \frac{r_2}{r_1} u'(c_2^*) \right]$. These results

Figure 2.1: Optimal consumption
 For a constant realization of λ For a constant realization of r_2



for optimal consumption, storage, and liquidation are summarized as

$$u'(c_1^*) = \begin{cases} u'(c_2^*) & \text{for low } (\lambda, r_2), & g_1^* > 0 \\ \in \left[u'(c_2^*), \frac{r_2}{r_1} u'(c_2^*) \right] & \text{for moderate } (\lambda, r_2), & g_1^* = a_1^* = 0 \\ \frac{r_2}{r_1} u'(c_2^*) & \text{for high } (\lambda, r_2), & a_1^* > 0. \end{cases} \quad (2.9)$$

A comparison of optimal consumption for early and late consumers is illustrated in the two diagrams in Figure 2.1 for variations in the realization of r_2 , for a constant realization of λ , and for variations in the realization of λ , for a constant realization of r_2 , respectively.

2.2. Digital currency

There is free entry of competitive, risk neutral banks and firms at $t = 0$ who have no endowment. Because of free entry, banks maximize their depositors' expected utility. Firms maximize expected profits in terms of consumption at $t = 2$. At $t = 0$, there is a quantity M_0^Y of digital currency, which is outside money and acts as numeraire, and hence the nominal unit of account in the economy.

Goods market There is a goods market at each date $t \in T_{0,1,2}$ with price P_t . Goods can be sold (q_t) at $t \in T_{0,1,2}$, stored (g_t) at $t \in T_{0,1}$, consumed (c_t) at $t \in T_{1,2}$, and invested as risky, illiquid assets (a_0) at $t = 0$, while assets (a_1) can be liquidated at $t = 1$. Lowercase letters denote real variables, and uppercase letters denote nominal variables with the digital currency unit of account.

Banks and firms There is a large number of ex-ante identical banks indexed by $j \in J$. At $t = 0$, bank j takes deposits D_0 from consumers, holds M_0^B in digital currency, and lends L_0^F to firms, where $L_0^F = D_0 - M_0^B$ is inside money created by the bank. Consumer deposits are based on selling goods at $t = 0$, $D_0 = g_C P_0$. Firms use loans to buy goods at $t = 0$ for storage (g_0) and investment in assets (a_0), $L_0^F = (g_0 + a_0)P_0$.

Depositors at bank j that have a realization as late consumers are indexed by $i \in I$ and withdraw a fraction $w^{ij} \in [0, 1]$ of D_0 at $t = 1$ and $(1 - w^{ij})$ of D_0 at $t = 2$. At $t = 1$, firms rollover an amount of borrowing L_1^{Fj} from bank j .¹² Bank j lends $L_1^{Bj'j}$ to $j' \in J$, which reflects bank j lending to other banks $j' \neq j \in J$ in the interbank market.

Nominal returns are $\delta_t^{kj} R_t^{kj}$ paid at $t \in \{1, 2\}$ on the contract type $k \in K \equiv \{D, F, Bj'\}$, which corresponds to deposits, loans to firms, and interbank loans to bank $j \neq j'$, respectively, and where $\delta_t^{kj} \leq 1$ is the fraction paid of the obligated contracted return R_t^{kj} . For example, deposits at bank j pay a return $\delta_t^{Dj} R_t^D$ when withdrawn at $t \in T_{1,2} \equiv \{1, 2\}$. If $\delta_t^{kj} < 1$, there is a default at date t by the agent paying the return, which requires paying all revenues to maximize δ_t^{kj} , where a bank's deposits are senior to interbank loans at $t = 2$. In addition, bank j must pay digital currency if demanded for withdrawals or else default. Since the aggregate state (λ, r_2) and depositor type i are not verifiable, there are incomplete contracts at $t = 0$ in the form of standard short-term debt. Nominal returns on bank deposits and loans made at $t = 0$, R_1^D , R_2^D , and R_1^F , are not contingent on the consumer's type i , bank's type j , or the realized state (λ, r_2) .

Bank liquidity shocks At $t = 1$, bank j has a realized and observable but not verifiable idiosyncratic liquidity shock of λ^j early consumers, where $\lambda^j \in (0, 1)$ and $E[\lambda^j] = \lambda$. A bank's idiosyncratic liquidity shock type is a random variable with binomial support. A fraction $\theta < \frac{1}{2}$ of banks have a 'high' liquidity shock realization, which for simplicity of notation have high type denoted by $j = h$, and have $\lambda^h = \lambda + e$, where $e \in (0, 1 - \lambda)$ is the higher fraction of early consumer withdrawals at bank h . The remaining fraction $(1 - \theta)$ of banks have a 'low' liquidity shock realization, with low type denoted as $j = l$. Since $E[\lambda^j] = \lambda$, $\lambda^l = \lambda - \frac{\theta}{1-\theta}e$.

¹²Firms can only borrow from one bank j at $t = 0$ and cannot borrow from other banks $j \neq j$ at $t = 1$, following from Rajan (1992) and Diamond and Rajan (2001). Bank j establishes an individual banking relationship with each firm it lends to at $t = 0$, which enables the bank to fully collect at $t \in \{1, 2\}$ on its loans to the firm without absconsion by the firm. A firm can develop such an absconsion-proof banking relationship with only one bank. The loan contract under such a banking relationship also disallows the firm from entering into a new banking relationship to borrow from a different bank $j' \neq j$ at $t = 1$.

‘Excess liquidity’ is defined as the amount of liquidity a bank has in excess of its expected early consumer withdrawals at $t = 1$, $(\lambda^j - \lambda)D_0R_1^D$. Excess liquidity represents a bank’s liquidity available for rolling over loans to its firms in excess of its expected rollovers and for interbank lending. A bank with a low liquidity shock, $j = l$, and its firms, are referred to as ‘liquid,’ because the bank has a positive amount of excess liquidity, $L_1^{Bl} \equiv \frac{\theta}{1-\theta}eR_1^D$. A bank with a high liquidity shock, $j = h$, and its firms, are referred to as ‘illiquid,’ because the bank has a negative amount of excess liquidity, $L_1^{Bh} \equiv -eR_1^D$. The net aggregate excess liquidity in the banking system is zero, since liquid banks in aggregate have excess liquidity of θeR_1^D and illiquid banks in aggregate have excess liquidity of $-\theta eR_1^D$.

Consumption For an early consumer at bank $j \in J$, consumption is denoted as c_1^j . For a late consumer $i \in I$ at bank $j \in J$, consumption from goods bought at $t = 1$ paid for using early withdrawals is denoted as c_1^{ij} ; and consumption from goods bought at $t = 2$ paid for using late withdrawals, as well as using any digital currency that is withdrawn early and stored at $t = 1$, is denoted as c_2^{ij} . These consumption amounts are expressed as

$$\text{early consumer} \quad t = 1: \quad c_1^j = \frac{\delta_1^{Dj} D_0 R_1^D}{P_1} \quad (2.10)$$

$$\text{late consumer} \quad t = 1: \quad c_1^{ij} = \frac{w^{ij} \delta_1^{Dj} D_0 R_1^D - M_1^{ij}}{P_1} \quad (2.11)$$

$$\text{late consumer} \quad t = 2: \quad c_2^{ij} = \frac{(1-w^{ij})\delta_2^{Dj} D_0 R_2^D + M_1^{ij}}{P_2}, \quad (2.12)$$

where for a late consumer $i \in I$ at bank $j \in J$, $M_1^{ij} \leq w^{ij} \delta_1^{Dj} D_0 R_1^D$ is the amount of the late consumer’s early withdrawal that is received in digital currency and stored for buying goods at $t = 2$.

Bank contracts Bank $j \in J$ offers the deposit contract (R_1^D, R_2^D) for deposits D_0 to consumers and the loan contract (R_1^F) for loans L_0^F to firms. Because of free entry by banks and firms at $t = 0$, bank j offers the deposit and loan contracts is to maximize its depositors’ expected utility,

$$EU^{ij} = E[\lambda u(c_1^j) + (1 - \lambda)u(c_1^{ij} + c_2^{ij})], \quad (2.13)$$

as follows:

$$\max_{\{R_1^D, R_2^D, R_1^F\}} EU^{ij} \quad (2.14)$$

$$\text{s.t. } t = 0: L_0^F + M_0^B \leq D_0 \quad (2.15)$$

$$t = 1: (\lambda^j + \sum_{i \in I} w^{ij}) D_0 \delta_1^{Dj} R_1^D \leq L_0^F \delta_1^{Fj} R_1^F - L_1^{Fj} - \sum_{j' \in J} L_1^{Bj'j} + M_0^B \quad (2.16)$$

$$t = 2: (1 - \lambda^j - \sum_{i \in I} w^{ij}) D_0 \delta_2^{Dj} R_2^D \leq L_1^{Fj} \delta_2^{Fj} R_2^{Fj} + \sum_{j' \in J} L_1^{Bj'j} \delta_2^{Bj'j} R_2^{Bj'j} + M_0^B \quad (2.17)$$

$$M_1^{ij} \leq M_0^B, \quad (2.18)$$

where M_0^B is digital currency stored by the bank from deposits at $t = 0$.

Firm optimization A firm borrowing from bank j consumes c_2^{Fj} at $t = 2$ and maximizes profit in the form of expected consumption, $E[c_2^{Fj}]$, as follows:

$$\max_{Q^{Fj}} E[c_2^{Fj}] \quad (2.19)$$

$$\text{s.t. } t = 0: (g_0 + a_0) P_0 \leq L_0^F \quad (2.20)$$

$$t = 1: L_0^F \delta_1^{Fj} R_1^F \leq L_1^{Fj} + q_1^j P_1 \quad (2.21)$$

$$t = 2: L_1^F \delta_2^{Fj} R_2^{Fj} \leq q_2^j P_2 \quad (2.22)$$

$$a_1^j \leq a_0 \quad (2.23)$$

$$q_1^j \leq g_0 + a_1^j r_1 - g_1^j \quad (2.24)$$

$$q_2^j \leq g_1^j + (a_0 - a_1^j) r_2 - c_2^{Fj} \quad (2.25)$$

$$a_1^j \geq 0 \quad (2.26)$$

$$g_1^j \geq 0, \quad (2.27)$$

where q_t^j is the quantity of goods sold by the firm at $t \in T_{1,2}$ and

$$Q^{Fj} \equiv \{a_0, a_1^j, \{g_t^j\}_{t \in T_{0,1}}, \{q_t^j\}_{t \in T_{1,2}}\}. \quad (2.28)$$

Liquidity strategy The *withdrawal strategy* for late consumer $i \in I$ at bank $j \in J$ is

$$\sigma^{ij} \equiv \{w^{ij}, M_1^{ij}\}, \quad (2.29)$$

while the *interbank lending strategy* for a liquid bank $j = l \in J$ is

$$\sigma^{Bj} \equiv \{L_1^{Bj'j}, R_2^{Bj'j}\}_{j' \in J} \quad (2.30)$$

The joint set of withdrawal strategies for all late consumers $i \in I$ at all banks $j \in J$ and interbank lending strategies for liquid banks $j = l \in J$ is defined as the *liquidity strategy* set $\sigma \equiv \{\sigma^{ij}, \sigma^{Bj}\}_{i \in I, j \in J}$.

Outside money The model is developed to allow for a parsimonious representation of outside money, whether in the form of traditional fiat money, public digital currency, or private digital currency. The case of fiat money, which I refer to as a regime without a public digital currency, is the same as the case of public digital currency, with the exception that fiat money cannot be held as outside money by consumers in the form of paper currency outside of the banking system for transactions. This is motivated by the fact that it is much too costly for paper currency to be stored, secured, and transacted in markets on the order of trillions of dollars that is transacted in the economy through electronic bank payments. Whereas, this potential for digital currency transactions outside of the banking system is the key distinguishing feature of digital currency studied in the model. This feature allows for potential bank runs from late consumers withdrawing and hoarding digital currency at $t = 1$ used to buy goods at $t = 2$.

The outside money supply of digital currency, M_0^Y , can take a strictly positive or zero amount. With a zero supply of outside money, the digital currency is still able to act as the nominal unit of account for deposits, loans, and prices in the goods market. This continues to enable a positive supply of inside money created by banks at $t = 0$ in the form of deposits, D_0 , denominated in the digital currency as unit of account.

Fiat money and digital currency alike have no inherent value to support it as a store of value for a positive outstanding net supply as outside money. In order to enable a positive store of value, I allow for a natural interpretation of fiat money or public digital currency to be backed by liquid assets. This allows for the central to create at $t = 0$ a positive supply of outside money which can be stored as reserves by banks for the case of fiat money, and as reserves as well as directly by consumers for the case of public digital currency, where bank withdrawals can take the form of depositors storing digital currency and later buying goods using digital currency payments in addition to depositors buying goods with bank deposits directly.

To maintain its a positive value of money and prevent hyperinflation of the price

level, if demand for such money were ever to fall, the central bank has to be able to redeem and retire the outside quantity. In an infinite-period model, as in reality, a central bank may never be required to redeem money. In a finite-period model, such as this one, a simplifying device is for the central bank to eventually redeem the outside money supply by the last period of the model.¹³

A positive supply of outside money, whether fiat or public digital currency, is created at $t = 0$ as a liability to the central bank, $M_0^Y > 0$, to buy goods g_0^Y , which are the liquid assets backing the digital currency supply. In order for the backing of the digital currency liability quantity to be credible, the central bank is able to commit to redeem and retire all of its digital currency liability quantity by acquiring it back through selling goods q_t^Y at $t \in T_{1,2}$. The central bank has consumption c_2^Y if it has remaining goods at $t = 2$. To enable an elastic price level, I assume that the central bank does not use its monopoly power as issuer to set the price level. Rather, the central bank acts as a competitive price taker at $t \in T_{0,1,2}$, which is represented by the maximization of expected its consumption as follows:

$$\max_{Q^Y} E[c_2^Y] \quad (2.31)$$

$$\text{s.t.} \quad t = 0: M_0^Y \leq g_0^Y P_0 \quad (2.32)$$

$$t = 1: q_1^Y P_1 \leq -M_1^Y \quad (2.33)$$

$$t = 2: q_2^Y P_2 \leq -M_2^Y \quad (2.34)$$

$$M_0^Y + M_1^Y + M_2^Y \leq 0 \quad (2.35)$$

$$q_1^Y \leq g_0^Y - g_1^Y \quad (2.36)$$

$$q_2^Y \leq g_1^Y - c_2^Y, \quad (2.37)$$

where

$$Q^Y \equiv \{\{M_t^Y\}_{t \in T_{0,1,2}}, \{g_t^Y\}_{t \in T_{0,1}}, \{q_t^Y\}_{t \in T_{1,2}}\}, \quad (2.38)$$

and M_t^Y is the quantity of digital currency the issuer supplies at $t \in T_{0,1,2}$. A negative supply $M_t^Y < 0$ is the redemption of the digital currency at $t \in T_{1,2}$.

In order to focus on examining the potential efficiency and financial stability benefits

¹³Alternatively, in the absence of backing with liquid assets, a positive supply of digital currency could only sustain a positive value if digital payments were not instantaneous, leading to a traditional transactions-based liquidity premium for outside money. With this assumption, the value of the outside digital currency stock would equal the NPV of future digital payment liquidity services for transactions.

of digital currency, as with fiat money, that derive from its role as an efficient means of payment and unit of account, electronic transaction payments from bank withdrawals or with the digital currency occur simultaneously within a period. I assume simultaneous digital transaction payments in order to shut down the channel for the digital currency to be held or have a positive value purely coming from a direct means-of-payment liquidity premium channel. A liquidity premium value for an outside money is based on value of holding outside money to use for lower payments transactions costs than using other assets, including inside money, if they are less liquid than outside money. Such a liquidity premium value for an outside money is equal to the present value of future payment liquidity services for non-instantaneous time for transactions. Efficient payments implies a lower liquidity premium value for outside money. With the simplification of assuming instant transaction bank or digital currency payments, there is no liquidity premium value. As such, for the case of a private digital currency with a positive outside quantity, I make an ad hoc representation that it is redeemed in a similar manner to a fiat or public digital currency supply for the purpose of allowing it to have a positive value in a finite-period model.

2.2.1. Equilibria

Definition 1. A market equilibrium is defined as prices $\{P_t\}_{t \in T_{0,1,2}} > 0$ such that at the optimizing quantities for firms $\{Q^{Fj}\}_{j \in J}$ and the issuer Q^Y , markets clear for goods at $t \in T_{0,1,2}$:

$$\begin{aligned} t = 0: & \quad g_0 + a_0 + g_0^Y = g_C, \\ t = 1: & \quad \sum_{j \in J} (c_1^j + \sum_{i \in I} c_1^{ij}) = \sum_{j \in J} q_1^j + q_1^Y, \text{ and} \\ t = 2: & \quad \sum_{j \in J} \sum_{i \in I} c_2^{ij} = \sum_{j \in J} q_2^j + q_2^Y. \end{aligned}$$

Proposition 2. For any liquidity strategy σ , there exists a market equilibrium, in which prices are given by

$$P_1 = \frac{\sum_{i \in [\lambda, 1]} \sum_{j \in J} (\lambda^j + w^{ij}) D_0 \delta_1^{Dj} R_1^D}{\sum_{j \in J} q_1^j} \quad (2.39)$$

$$P_2 = \frac{\sum_{i \in [\lambda, 1]} \sum_{j \in J} (1 - \lambda^j - w^{ij}) D_0 \delta_2^{Dj} R_2^D}{\sum_{j \in J} q_2^j}. \quad (2.40)$$

Equilibrium prices at $t = 1$ and $t = 2$ are reflected as the amount of money supplied by consumers for purchasing goods in a period divided by the amount of goods sold by

firms. If $M_0^Y = 0$, the price level at $t = 0$, P_0 , is not determined and, for simplicity, is normalized to one.

The seniority of bank deposits over interbank loans implies that a bank that defaults at $t = 2$ maximizes paying obligated returns on deposits before paying any returns on interbank loans. This seniority rule arises as part of the optimal bank contract between bank j with its depositors at $t = 0$ to ensure that other banks $j' \neq j$ cannot lend to bank j at $t = 1$ in order to expropriate late consumers withdrawing from bank j at $t = 2$. For simplicity, I assume that there is a pro rata sharing rule for defaulted returns among deposits withdrawn at either date $t \in T_{1,2}$, $\delta_t^{Dj} < 1$, and among interbank loans at $t = 2$, $\delta_2^{Bj'j} < 1$. Results do not change if there is any type of priority rule instead of a pro rata sharing rule, such as for a sequential service constraint for deposit withdrawals at $t = 1$ in which some deposit withdrawals had no default, $\delta_1^{Dj} = 1$, and the remaining deposit withdrawals had a complete default, $\delta_1^{Dj} = 0$.

A default by bank j on deposit withdrawals at $t = 1$, $\delta_1^{Dj} < 1$, requires the bank to pay all of its revenues at $t = 1$ for withdrawals. This implies that the bank cannot rollover any lending to its firms, $L_1^{Fj} = 0$, or lend to other banks. Hence, the bank will not have any revenues at $t = 2$, has a complete default at $t = 2$, $\delta_2^{Dj} = 0$, and cannot borrow from other banks or the issuer at $t = 1$, $L_1^{Bj'j} = 0$ for all $j' \in J$. Such a bank is referred to as liquidated at $t = 1$, since it has no loan assets after $t = 1$. Since the banks' firms cannot rollover any of their loans, these firms will also default at $t = 1$, $\delta_1^{Fj} < 1$. The firms must fully liquidate their assets, $a_1^j = a_0$, to sell goods and repay as much of their loans at $t = 1$ as possible. Depending on the amount of a bank's withdrawals at $t = 1$, the bank may be able to avoid default by rolling over a limited amount of loans to its firms at $t = 1$. These firms may be need to liquidate a part of their assets to avoid defaulting on the amount of loans not rolled over. However, if the firms cannot avoid default at $t = 1$, even by liquidating all of their assets, the bank will not rollover any amount of loans, since the firms would have a complete default at $t = 2$, δ_2^{Fj} .

Definition 2. *A Nash equilibrium of the strategic liquidity game is defined as*

$$\{\sigma^{ij}, \sigma^{Bj} | \{\sigma^{i'j'}, \sigma^{Bj'}\}_{i' \in I, j' \in J}\}_{i \in I, j \in J},$$

which is the set of withdrawal strategies σ^{ij} for each late consumer $i \in I$ at each bank $j \in J$, and interbank lending strategies σ^{Bj} for each bank $j \in J$, which are each a best response given $\{\sigma^{i'j'}, \sigma^{Bj'}\}_{i' \in I, j' \in J}$, the withdrawal strategies of all other late consumers

$i' \in I$, and interbank lending strategies of all other banks $j' \in J$.

2.2.2. Optimal equilibrium

An elastic value of a digital currency, whether privately or publicly issued, can enable the optimal allocation of the full-information first best, with financial stability of no bank defaults, for any realization of the aggregate state (λ, r_2) and size of bank liquidity shocks (e) .

Lemma 1. *Optimal interbank lending requires multiple liquid banks to lend to each illiquid bank. The optimal interbank lending by each liquid bank is to lend its excess liquidity, such that each illiquid bank can optimally borrow on the interbank market the amount of its shortage of excess liquidity:*

$$\text{liquid banks:} \quad \sum_{h \in J} L_1^{Bhl} = L_1^{Bl} \quad \forall e \in E, j \in J \quad (2.41)$$

$$\text{illiquid banks:} \quad \sum_{l \in J} L_1^{Blh} = L_1^{Bh} \quad \forall e \in E, j \in J. \quad (2.42)$$

Without idiosyncratic bank liquidity shocks, $e = 0$, banks would choose to require the firms they lend to store the optimal amount of real liquidity at $t = 0$, $g_0 = g_0^*$. The optimal storage follows from the Euler equation in the banks' optimization of their depositors' expected utility for the provision of optimal liquidity for early consumers following that is equivalent to the first best, $E[u'(c_1^*)] = E[r_2 u'(c_2^*)]$. The expected interbank market rate is equal to the expected return on assets, \bar{r}_2 , which is greater than the implicit expected return paid on returns on deposit withdrawals at $t = 2$ relative to $t = 1$: $\frac{E[c_2^*]}{E[c_1^*]} < \frac{E[u'(c_1^*)]}{E[u'(c_2^*)]} = \bar{r}_2$. Since there is no interbank borrowing and lending required, there is no distortion to consumption among depositors at different banks. However, in the general case that includes positive idiosyncratic bank liquidity shocks, $e > 0$, the need for interbank borrowing and lending requires an expected interbank rate equal to the expected return on relative deposit withdrawals, $\frac{E[c_2^*]}{E[c_1^*]} < \bar{r}_2$. This leads to the incentive for banks to free-ride off of borrowing an expected positive amount of liquidity in the interbank market rather than requiring their firms to hold the optimal liquidity g_0^* at $t = 0$. Thus, liquidity regulation for bank lending of $g_0 = g_0^*$ is required for any potential optimal provision of liquidity in the economy.¹⁴

¹⁴See Diamond and Rajan (2001, 2005, 2006) for a further developed model on bank lending to firms and optimal liquidity provision.

Proposition 3. *For all realizations of r_2 and λ , and all bank liquidity shocks e , there exists a Nash equilibrium with optimal interbank lending, and without bank runs,*

$$w^{ij} = 0 \quad \forall i \in I, j \in J. \quad (2.43)$$

Moreover, there is a unique market equilibrium. Consumption is the optimal first best, and there are no bank defaults:

$$c_1^j = c_1^* \quad \forall j \in J \quad (2.44)$$

$$c_2^{ij} = c_2^* \quad \forall i \in I, j \in J \quad (2.45)$$

$$c_1^{ij} = 0 \quad \forall i \in I, j \in J \quad (2.46)$$

$$\delta_t^{kj} = 1 \quad \forall j \in J, k \in K, t \in T_{1,2}. \quad (2.47)$$

Withdrawing and hoarding currency at $t = 1$ is a weakly dominated strategy. Late consumers are indifferent to buying goods later with currency or immediately with digital payments. Thus, late consumers do not prefer to withdraw early and runs do not materialize. Prices are given by

$$P_1 = \frac{\lambda D_0 R_1^D}{q_1} \quad (2.48)$$

$$P_2 = \frac{(1-\lambda) D_0 R_2^D}{q_2}. \quad (2.49)$$

Since deposits pay out nominal amounts, the bank can pay fixed promises in terms of the digital currency numeraire, yet depositors' consumption can flexibly respond to aggregate real and liquidity shocks in the economy through elastic prices. Hence, the real value of deposits are efficiently contingent on the aggregate state (λ, r_2) . Such elastic prices, which reflects an elastic value of the digital currency, also increases financial stability since banks do not have runs even when there are low real returns on illiquid assets or high liquidity needs by depositors.

An elastic value of a digital currency enables elastic prices in the economy, which supports a financial system that can create efficient asset and liquidity risk sharing in the economy through leveraged financial intermediaries. The market efficiently rations goods between early and late consumers due to the price mechanism. For a moderate realization of λ and r_2 , the equilibrium price levels at $t = 1$ and $t = 2$ are sufficiently close to one such that firms sell at $t = 1$ all of their goods stored from $t = 0$ and sell

at $t = 2$ the returns on all their assets. For a low enough joint realization of λ and r_2 , there is relative downward pressure on P_1 and upward pressure on P_2 . With fewer early consumers, the amount of inside money spent for goods is reduced at $t = 1$ and increased at $t = 2$. With lower returns, fewer goods produced by assets are available to sell at $t = 2$. Firms respond to these market prices by optimally storing some of their goods held at $t = 1$ to sell at $t = 2$, which provides for equal consumption among early consumers withdrawing at $t = 1$ and late depositors withdrawing at $t = 2$. Conversely, for a high enough joint realization of λ and r_2 , there is relative upward pressure on P_1 and downward pressure on P_2 . Firms optimally respond by optimally liquidating some of their assets to sell additional goods at $t = 1$. Moreover, for all realizations of (λ, r_2) , firms have zero profits, which reflects free entry and competition among firms.

The elasticity of the price level at $t = 1$ and $t = 2$ reflects the elastic value of a digital currency, since the real value of the digital currency at each period is the inverse of the price level. This elastic value of the digital currency that provides the optimal allocation of consumption also enhances financial stability against two primary risks inherent in the banking system. One is the threat of solvency-based banks runs from the potential insolvency of the banking system in the case of low real returns on assets. The second is the threat of liquidity-based bank runs from the potential of late consumers running on banks.

First, consider the threat of insolvency in the case of low realizations of r_2 . As explained above, P_2 increases due to the reduction in goods available to sell at $t = 2$. This leads firms to hold over goods from $t = 1$ to sell at $t = 2$, such that late consumers do not receive any greater consumption by running the bank to buy goods at $t = 1$. Moreover, banks are effectively hedged on their nominal deposit liabilities at $t = 2$. The equilibrium price level at $t = 2$ remains above one, which implies that the real cost of banks' $t = 2$ deposit liabilities falls enough that banks do not default.

Second, consider the case of a liquidity-based run, in which all late consumers run on the banking system by withdrawing and buying goods at $t = 1$. With asset liquidation by firms to sell additional goods at $t = 1$, P_1 would still be sufficiently greater than one that firms would not default on their loans to banks, and banks would not default on paying withdrawals. A marginal or discrete mass of late consumers could deviate by withdrawing instead at $t = 2$ to buy greater consumption than by withdrawing to buy goods. In the case of any or all late consumers running on a marginal individual bank or discrete mass of banks, the interbank market would receive sufficient excess liquidity

from the marginal increase in P_1 to lend to the banks with liquidity-based runs such that the banks do not default. Regardless of a systemic or marginal liquidity-based runs, late consumers would prefer to withdraw at $t = 2$. Hence, such liquidity-based runs do not occur in equilibrium.

3. Financial crises

With a private digital currency, in addition to the optimal market equilibrium with no banks, there are also two types of financial crisis equilibria. Section 4.1 shows that there is an equilibrium with an interbank market liquidity freeze, in which illiquid banks cannot borrow on the interbank market. As the size of the bank liquidity shock increases, there is an increase in bank defaults, excessive asset liquidation, runs on illiquid banks, and systemic contagion runs on liquid banks. Section 4.2 shows that there is also an equilibrium with withdrawal runs, in which late consumers withdraw and store currency at $t = 1$, which can lead to a complete banking collapse and liquidation in the entire economy. Section 4.3 shows that with a public digital currency, the central bank is able to elastically provide its digital currency in order to act as lender of last resort to prevent the interbank market freeze and withdrawal run equilibria.

3.1. Interbank market liquidity freeze

When the interbank market breaks down, liquid banks cannot lend and there is an interbank market freeze equilibrium. There is no interbank lending, which leads to inefficient liquidation by illiquid banks and price deflation. The liquidity of all banks is reduced, and all consumers have inefficient liquidity risk sharing. Larger bank liquidity shocks lead to runs at illiquid banks and eventually to systemic contagion runs at liquid banks as well.

Equilibrium prices conditional on the size of the bank liquidity shock, e , are denoted in the market equilibrium with an interbank freeze as $P'_t(e)$, and without a freeze as $P_t(e)$, for $t \in T_{1,2}$, where prices without a freeze are the optimal prices, not conditional on the size of the shock, $P_t(e) = P_t^*$ for $t \in T_{1,2}$. A freeze equilibrium does not require coordination runs by late consumers, although these types of runs would exacerbate the negative impacts of the interbank market freeze. Instead, the assumption is made that late consumers at a bank with liquidity shock of type $j \in \{h, l\}$ only withdraw early if they have greater consumption by buying goods at $t = 1$ than by withdrawing and

buying goods at $t = 2$: $\frac{\delta_1^{Dj} D_1}{P_1} > \frac{\delta_2^{Dj} D_2}{P_2}$, for $e \in E$, $j \in \{h, l\}$.

Lemma 2. *There exists an interbank market freeze equilibrium with no interbank lending:*

$$L_1^{Bj'j} = 0 \quad \forall e \in E, j' \in J, j \in J. \quad (3.1)$$

Price deflation For any size of the bank liquidity shock, e , illiquid banks have a default at $t = 2$, $\delta_2^h < 1$, in the absence of borrowing from liquid banks. This is caused because, due to negative excess liquidity from larger than expected early consumer withdrawals, illiquid banks cannot rollover the optimal lending to their firms at $t = 1$ in order to pay out their additional early consumer withdrawals without interbank borrowing. Their firms have to inefficiently liquidate an amount of illiquid assets in excess of the optimal amount, which leads to lower revenues that illiquid banks can receive on loan returns at $t = 2$ for paying their withdrawals at $t = 2$, leading to the $t = 2$ default, $\delta_2^h < 1$

In an interbank freeze, liquid banks not only have no incentive to lend to illiquid banks, they also lose their excess liquidity to be able to lend. In a partial equilibrium analysis with prices held fixed as $P_t(e) = P_t^*$, a liquid bank's excess liquidity based on lower than expected early consumer withdrawals is insufficient in the form of an interbank loan from an individual liquid bank to prevent an illiquid bank from defaulting on deposits at $t = 2$. Thus, any interbank loan from the liquid bank would incur a default, since interbank loans are junior to deposits.

Moreover, in general equilibrium with prices $P_t'(e)$, liquid banks lose the potential to make interbank loans because their excess liquidity falls to zero. The excess liquidation of assets by firms borrowing from illiquid bank provides these firms with additional revenues from the additional goods sold at $t = 1$ to repay the amount of loans it cannot roll over. These excess goods on the market leads to a lower price level, $P_1'(e)$. The lower price level implies that firms at liquid banks have lower revenues for their optimal goods sold on the market, q_1^{l*} . These firms would also have to liquidate excess assets in order not to default at $t = 1$ according to the optimal loan rollover amount L_1^{Fj*} . Since such liquidation would lead to lower returns that could be received at $t = 2$ on loans to firms, liquid banks use their excess liquidity to provide larger rollovers to their firms to avoid excess asset liquidation. The excess liquidity provided to firms implies that liquid banks lose any excess liquidity at $t = 1$ to lend to illiquid banks. Liquidity available for the

interbank market actually dries up.

In effect, when illiquid banks cannot borrow the excess liquidity of liquid banks as in the optimal interbank market, illiquid banks acquire this excess liquidity from liquid banks by capturing an increased share of funds available in the goods markets at $t = 1$. This is achieved through both illiquid banks' firms supplying more goods through asset liquidation, as well as the lower equilibrium price level $P_1'(e)$ that reduces the funds acquired by liquid banks' firms for the amount of goods they supply.

Debt deflation A result of price deflation at $t = 1$ is debt deflation for banks' deposit liabilities and firms' loan liabilities, which have opposite impacts on welfare. Lower prices increase the real cost of deposits for banks, which leads to bank defaults, reduced loan rollover by illiquid banks, and loss of excess liquidity for liquid banks. This loss of excess liquidity is due to the increase in the real cost of firms' loans, which requires greater loan rollovers for liquid firms. Illiquid firms have to liquidate assets not only because of reduced loan rollovers, but also because of asset liquidation required to meet the increased real cost of their loans that are not rolled over.

Despite these inefficiencies arising from the increase in real debt burdens, the elasticity of prices that allows for price deflation on net reduces the negative impact of the interbank market freeze on illiquid banks compared to if prices were not elastic. If the price level were to remain at P_1^* rather than decrease to $P_1'(e)$, illiquid firms would have to liquidate an even greater amount of assets to pay off their loans at $t = 1$, which would lead to even lower returns on loans paid to illiquid banks and greater bank defaults at $t = 2$.

However, debt deflation arising from price deflation is the mechanism that can lead to contagion runs at liquid banks for a large enough size of bank liquidity shocks, e . In addition, the drop in P_1 and increase in P_2 implies that consumers from all banks have inefficient ex-ante risk sharing. Late consumers from all banks consume less than the first best. Early consumers from all banks consume more than the first best for small shocks and less than the first best for large enough shocks.

Proposition 4. *In an interbank market freeze equilibrium with positive bank liquidity shocks, $e > 0$, there are bank defaults, excessive asset liquidation, price deflation, debt deflation, and suboptimal liquidity-risk sharing for consumers. As the size of bank liquidity shocks increase, the Nash equilibrium has runs at illiquid banks and systemic contagion runs at liquid banks.*

The extent of bank defaults, liquidation, price deflation, banks runs, and inefficient depositor liquidity-risk sharing depends on the size of the bank liquidity shock, e . The three regimes of bank liquidity shocks are defined for shock sizes that are relatively ‘small,’ $e' \in E' \equiv (0, e_0]$; ‘moderate,’ $e'' \in E'' \equiv (e_0, e_1]$; and ‘large,’ $e''' \in E''' \equiv (e_1, 1 - \lambda)$, where $0 < e_0 < e_1 < 1 - \lambda$.

Defaults by illiquid banks For small shocks, e' , there is only a partial default by illiquid banks at $t = 2$, $\delta_2^h < 1$. The shock is small enough that there are no runs on illiquid banks. A marginal late consumer at an illiquid bank still has higher consumption by withdrawing at $t = 2$ rather than at $t = 1$: $c_2^{ih} \geq c_1$. However, these late consumers have lower consumption than late consumers at liquid banks: $c_2^{ih} < c_2^{il}$. Thus, a new form of inefficient liquidity risk arises for late consumers caused by banks’ idiosyncratic liquidity risk. Moreover, liquidity-risk consumption sharing for consumers of all banks is suboptimal. The additional amount of goods from excess liquidation sold by firms at illiquid banks implies that the $t = 1$ price level falls below the optimal price: $P_1'(e') < P_1^*$. Early consumers at all banks benefit with higher consumption from more goods on the market at $t = 1$ reflected by the lower price level: $c_1^j > c_1^*$. However, this comes at the cost of lower consumption not just for late consumers at illiquid banks but also at liquid banks: $c_2^{il} < c_2^*$. There are fewer goods on the market at $t = 2$ because of the excessive liquidation by firms at illiquid banks, which leads to higher $t = 2$ prices than the optimum: $P_2'(e') > P_2^*$. Thus, consumption is suboptimal for all consumers.

Bank runs at illiquid banks For moderate shocks, e'' , there are complete bank runs at illiquid banks. For $e > e_0$, illiquid banks have to pay all revenues at $t = 1$ for early consumer withdrawals and cannot rollover any loans to their firms. This creates a complete liquidation of assets by firms borrowing from illiquid banks. Illiquid banks have no revenues and a complete default at $t = 2$, $\delta_2^h = 0$. This forces runs at illiquid banks, since late consumers must withdraw at $t = 1$, $w^{ih} = 1$. These forced runs compound the illiquidity problem at high-shock banks. Illiquid banks have a complete demand for withdrawals at $t = 1$, which these banks cannot meet and have a partial $t = 1$ default, $\delta_1^h < 1$.

Price deflation at $t = 1$ is greater than in the case of small e' shocks: $P_1'(e'') < P_1'(e')$. The increased amount of goods on the market at $t = 1$ coming from the liquidation by firms outweighs the increased amount of money to buy goods coming from late

consumers withdrawing at illiquid banks. Conversely, reduced amount of goods on the market at $t = 2$ outweighs the reduced money to buy goods, and $t = 2$ prices are even higher: $P_2(e'') > P_2(e')$. The consumption for all depositors at illiquid banks is lower than the optimal amount at $t = 1$ and that for early consumers at liquid banks: $c_1^{ih} = c_1^h < c_1^* < c_1^l$. Liquidity-risk sharing among depositors at liquid banks, and among all depositors, is distorted even further than in the case of small e' shocks.

Systemic contagion runs at liquid banks For large shocks, e''' , there is large enough debt deflation that causes contagion runs at liquid banks. Under large shocks, the price level falls enough at $t = 1$ such that in the absence of late consumers withdrawing from liquid banks at $t = 1$, their consumption c_2^{il} is lower than consumption c_1^l for early consumers at liquid banks. Hence, there is a partial run of liquid banks by late consumers, with $w^{il} > 0$. The amount of early withdrawals by late consumers at liquid banks rises until the consumption of depositors withdrawing at $t = 1$ falls to equal the consumption of depositors withdrawing at $t = 2$: $c_1^{il} = c_1^l = c_2^{il}$. Thus, there is suboptimal consumption risk among depositors at liquid banks as well as between depositors at illiquid and liquid banks.

When an initial individual bank shock leads the bank to liquidate loans, a less liquid market creates deflation and causes other banks' loans to lose value due to their own firms defaulting. Thus, illiquidity on the liability side of banks can spill over to illiquidity and loss of value on the asset side of other banks, which may cause illiquidity on the depositor side of these other banks as their depositors withdraw. Initial banks that fail due to illiquidity can lead to other banks failing due to a feedback loop between the liability and asset sides of the banking system.

Mitigating factors If banks were to make two-period loans to firms at $t = 0$, firms would default at $t = 1$ for large enough shocks, but banks could renegotiate their loans with firms to give the same results as with one-period loans. However, if illiquid firms could partially borrow from liquid banks, the extent of the banking crisis would be reduced for a given size banking liquidity shock, e . Furthermore, if firms could fully borrow as needed from additional banks at $t = 1$, the interbank freeze equilibrium would not exist. Illiquid firms could make up for the shortages on their rollovers from their banks by borrowing from liquid banks, which would preclude these firms from excess asset liquidation and the price level from falling at $t = 1$. Without price deflation, liquid

firms would not need additional rollovers from their liquid banks, which in turn can afford to lend their excess liquidity to illiquid firms. Liquid banks' excess liquidity not lent in the interbank market, for illiquid banks to lend on to their firms, is instead lent directly to the illiquid firms. In such a case, illiquid banks have no defaults at $t = 2$. Moreover, this implies that rather than lending to illiquid firms, any individual liquid bank can instead lend its excess liquidity to an illiquid bank without default, even in the absence of no other lending on the interbank market. Hence, in equilibrium, all liquid banks will lend their excess liquidity to illiquid banks, which precludes the interbank market freeze as an equilibrium.

3.2. Withdrawal runs

The second type of financial crisis equilibrium with a private digital currency is a complete run on the banking system based on withdrawal runs. Digital currency can be stored as outside money that can be used, equivalent to bank deposits as inside money, as an efficient means of payment to buy goods. This efficient means of payment enables late consumers to withdraw digital currency at $t = 1$ to buy goods at either $t = 1$ or $t = 2$.

However, the ability for digital currency to be withdrawn and held outside of banks allows for the potential of withdrawal runs on the entire banking system. In contrast to an interbank freeze, in which liquid banks do not lend to illiquid banks, withdrawal runs take the form of late consumers that withdraw digital currency as outside money at $t = 1$. Excessive withdrawals of digital currency can deplete and disintermediate the entire banking system, in which all banks become illiquid and default. Bank defaults from withdrawal runs lead to excessive asset liquidation, further bank defaults, and complete bank failures. Bank failures can be further compounded by downward spiraling price deflation and debt deflation.

Proposition 5. *With a digital currency regime, there exists a withdrawal run Nash equilibrium in the form of digital currency withdrawals by late consumers that creates a complete run and liquidation of the banking system.*

3.3. Elastic digital currency

The interbank freeze and withdrawal run equilibria give rise to the role for a lender of last resort. In these equilibria, defaults and failures occur for banks that are otherwise

fundamentally solvent but are illiquid due to excessive withdrawals at $t = 1$, whether from a larger amount of early consumers not met with interbank borrowing in the case of interbank freezes, or from digital currency withdrawals from late consumers. The digital currency issuer has the sole ability to create an additional quantity of the supply of outside money, which gives the issuer a natural monopoly over the outside supply of liquidity available to banks. Because of this, the issuer has the unique ability to act as a potential lender of last resort by issuing an additional quantity of digital currency that is lent to illiquid banks at $t = 1$.

The creation of additional digital currency at $t = 1$ represents an elastic supply of the quantity of digital currency as outside money, which is distinct from the elasticity of prices. Even with an inelastic supply of the digital currency, as is the case in the optimal equilibrium in which additional digital currency is not issued at $t = 1$, prices are elastic because they are denominated in the digital currency as a nominal unit of account. Such an inelastic digital currency permits the optimal equilibrium, even with the realization of low asset returns and high early consumer liquidity needs, because of elastic prices. However, an inelastic digital supply of the digital currency also permits the interbank freeze and withdrawal run financial crises equilibria, which elastic prices do not prevent and can even exacerbate.

However, the issuer can elastically supply its digital currency by issuing an additional amount at $t = 1$ that enables the issuer to act as lender of last resort to illiquid banks. To consider whether the issuer is able to prevent financial crises equilibria as a lender of last resort, the issuer optimization is slightly updated in two ways. First, under a fiat or public digital currency, the central bank's objective function is to maximize depositors' expected utility: $EU^{CB} = EU^{ij}$. Second, the issuer can make loans to illiquid banks according to an update to its budget constraints at $t \in T_{1,2}$:

$$t = 1: q_1^Y P_1 \leq -M_1^Y + \sum_{j \in J} L_1^{BjY} \quad (3.2)$$

$$t = 2: q_2^Y P_2 \leq -M_2^Y - \sum_{j \in J} L_1^{BjY} \delta_2^{BjY} R_2^{BjY}, \quad (3.3)$$

where the bank *lending strategy* for the issuer is

$$\sigma^{Yj} \equiv \{L_1^{BjY}\}_{j \in J} \quad (3.4)$$

The set of banks J that a bank $j \in J$ can lend to is updated to $J^Y \equiv J \cup Y$, which

reflects bank j borrowing ($L_1^{BYj} < 0$) from the issuer $j' = Y$ as well as lending to other banks $j' \neq j \in J$ in the interbank market. A Nash equilibrium of the strategic liquidity game is updated as $\{\sigma^{ij}, \sigma^{Bj}, \sigma^{Yj} | \{\sigma^{i'j'}, \sigma^{Bj'}, \sigma^{Yj'}\}_{i' \in I, j' \in J}\}_{i \in I, j \in J}$ to include the issuer bank lending strategy.

The additional outside money created at $t = 1$ is backed by the issuer's claim on the banks it lends to. Regardless of the seniority of the issuer's loans to banks, the issuer can create and lend large enough amounts to illiquid banks to ensure they do not default at $t = 1$ and $t = 2$. Hence, the issuer does not face any risk of banks defaulting on the loans, which makes the loans safe assets backing the issuer's additional quantity of digital currency. Borrowing banks repay the loans, comprised of outside digital currency at $t = 1$, in kind at $t = 2$ with outside money, which is thus redeemed by the issuer at $t = 2$ without needing to utilize the goods market for redemption of the additional outside money issued at $t = 1$.

Proposition 6. *Under a fiat money or public digital currency regime, the central bank acts as lender of last resort by providing an elastic outside money supply. The interbank freeze equilibrium and withdrawal run equilibrium do not exist. The optimal market equilibrium and Nash equilibrium without bank runs are unique. Whereas, with a private digital currency regime, there is no lender of last resort, and the interbank freeze and withdrawal run equilibria are unchanged.*

With a public digital currency, financial crises do not occur because the central bank is able and willing to elastically expand the quantity of outside money to act as lender of last resort to banks that become illiquid at $t = 1$. The interbank market freeze equilibrium does not occur because if liquid banks were to not lend to illiquid banks, the illiquid banks would borrow from the central bank and have no defaults. In such case, illiquid banks rollover the optimal amount to their firms, and there is no excessive asset liquidation by illiquid firms. Without excessive goods on the market at $t = 1$, there is no price deflation or debt deflation. Liquid banks do not need to provide their excess liquidity as larger rollovers to their firms. On an individual marginal basis as well as collective basis, liquid banks prefer to lend to illiquid banks. These interbank loans do not have defaults and provide a greater return than returns that liquid banks could obtain from larger rollovers to firms, which reflects the more efficient use of liquid banks excess liquidity. Moreover, liquid banks require the larger returns available from interbank loans in order not to default on their own larger-sized depositor withdrawals

that occur at $t = 2$ from their larger share of late consumers.

The withdrawal runs equilibrium does not occur because if late consumers were to withdraw digital currency to store as outside money at $t = 1$, banks would again borrow from the central bank and have no defaults. In such case, on an individual marginal basis as well as collective basis, late consumers prefer to withdraw at $t = 2$, rather than at $t = 1$, for a greater return on deposits and greater consumption.

Hence, in the consideration of either the case of a potential interbank market freeze or potential withdrawal run, the result is that with a central bank as the digital currency issuer, banks can borrow for their full liquidity shock needs on the interbank market and do not have withdrawals by late consumers at $t = 1$. Thus, in equilibrium, banks do not borrow from the central bank. The potential cases of an interbank freeze and withdrawal run are out-of-equilibrium threats that are prevented from occurring as equilibria because of the ability and willingness of the central bank to elastically supply its digital currency as lender of last resort.

However, with a private digital currency, these types of financial crises do occur as equilibria for two reasons. First, a private issuer does not choose to elastically supply its digital currency and act as lender of last resort. In either an interbank market freeze or withdrawal run, the digital currency issuer makes a profit if a crisis equilibrium occurs by not lending to illiquid banks and preventing the crisis. A private issuer chooses to make such a profit by allowing a crisis to occur. While a private issuer does not cause a crisis equilibrium to occur, the private issuer allows for it and makes a profits when it happens. Whereas, the public issuer foregoes making a profit in order to prevent crises. Second, with a private digital currency, the central bank is not able to act as a lender of last resort to prevent crises. Since the central bank is not able to create the private digital currency, it does not have access to an elastic supply required for the ability to sufficiently lend to illiquid banks for the prevention of a crisis.

4. Concluding remarks

A major theme in the academic literature since the financial crisis is investigating causes of fragility in the leveraged financial system. Now, with the heightened interest and concern about potential impact on the financial system that may come from fintech, understanding the financial fragility that major financial technologies may bring is crucial.

This paper provides an initial examination within the burgeoning literature on fintech

on the potential impact of digital currency on the banking system. A digital currency would not displace the banking system, which is resilient from aggregate return and liquidity risk with an elastic price level under a digital currency as with fiat money, but is subject to interbank market lending freezes exacerbated by an elastic price level in the economy.

However, a primary distinction about the stability of the financial system arises based on whether the digital currency within the economy is privately issued, such as bitcoin, or is publicly issued by a central bank, which is a growing consideration by central banks worldwide. In the case of a public digital currency, the central bank can elastically supply its own digital currency, as with fiat money, which enables the central bank to act as the lender of last resort. This prevents financial crises from occurring and provides a unique equilibrium with the first best allocation of liquidity and consumption.

However, for an economy under the regime of a private digital currency, the private issuer does not provide an elastic supply of its digital currency to act as lender of last resort. The inelastic supply of the private digital currency allows for financial crisis equilibria from freezes in the interbank market and withdrawal runs of digital currency that is depleted from the banking system. In such a regime, the central bank cannot create the additional digital currency required for the central bank to act as lender of resort to prevent these crises.

Appendix: Proofs

Proof for Proposition 1. The optimization of equations (2.1)-(2.4) gives binding budget constraints and first order conditions for EU^i with respect to a_0 , which gives equation (2.5); g_1 , which gives the first equation of (2.9); and a_1 , which gives the third equation of (2.9). Binding budget constraints imply equations (2.6)-(2.8).

Proof for Proposition 2. Market clearing for goods at $t \in T_{0,1,2}$ requires that all constraints bind for the optimizations of firms given by equations (2.19)-(2.25) and the issuer given by equations (2.31)-(2.37), with the possible exception of the firm constraint $a_1^j \leq a_0$ in equation (2.23). Necessary first order conditions and sufficient second order conditions hold for $\{Q^{Fj}\}_{j \in J}$ in the firm optimization and Q^Y in the issuer optimization. Thus, the market equilibrium $(P_t)_{t \in T_{1,2}}$ exists and is unique up to an indeterminate price level at $t = 0$, P_0 . Market clearing in the goods market at $t = T_{1,2}$ implies the equilibrium price P_t solved as equations (2.39) and (2.40) from binding budget constraints in the

optimizations of the firm, issuer, as well as of the bank given by equations (2.15)-(2.17) and consumption given by equations (2.10)-(2.12).

Proof for Lemma 1. Substituting into the $t = 1$ budget constraint (2.16) of bank $j \in \{h, l\}$ for $w^{ij} = 0$, $\delta_1^{Dj} = 1$, $R_1^D = \frac{g_0^*}{\lambda}$, $M_0^B = D_0 - L_0^F$, $L_0^F \delta_1^{Fj} R_1^F = L_1^{Fj} + q_1^j P_1$, $q_1^j = g_0^* + a_1^* r_1 - g_1^*$, $\lambda^h = \lambda + e$, and $\lambda^l = \lambda - \frac{\theta}{1-\theta} e$, and solving for $\sum_{j' \in J^Y} L_1^{Bj'j}$, where $L_1^{BYj} = 0$, gives equations (2.41) and (2.42). Since $\theta < \frac{1}{2}$, the amount each liquid bank $j = l$ lends, $\frac{\theta}{1-\theta} e R_1^D$, is less than the amount that each illiquid bank $j = h$ borrows, $e R_1^D$, each illiquid bank borrows from more than one liquid bank.

Proof for Proposition 3. Consider a liquidity strategy set σ with optimal interbank lending, $\sigma^{Bj} = L_1^{Bj}$ for $j \in \{h, l\}$, and without early withdrawals, $w^{ij}(\lambda, r_2) = 0$, which for feasibility requires $M_1^{ij}(\lambda, r_2) = 0$, for all $\lambda \in (0, 1)$, $r_2 \in (0, r_2^{\max})$, $e \in [0, 1 - \lambda)$, $i \in I$, $j \in J$. Substituting into market equilibrium prices, there are no bank defaults, and consumption equals the first best allocation, given by equations (2.6)-(2.8). A late consumer's consumption at $t = 2$ is $c_2^{ij} = \frac{D_0 R_2^D}{P_2}$. Suppose there is any deviation in the withdrawal strategy $\sigma^{i''j''}$ by any late consumer i'' at any bank j'' , such that $w^{i''j''}(\lambda, r_2) > 0$ and $M_1^{i''j''}(\lambda, r_2) \leq w^{i''j''}(\lambda, r_2)$ for any $\lambda \in (0, 1)$, $r_2 \in (0, r_2^{\max})$, and $e \in [0, 1 - \lambda)$. This late consumer's consumption is $c_1^{i''j''} + c_2^{i''j''}$, where $c_1^{i''j''} = \frac{w^{i''j''} \delta_1^{Dj} D_0 R_1^D - M_1^{i''j''}}{P_1}$, $c_2^{i''j''} = \frac{(1-w^{i''j''}) \delta_2^{Dj} D_0 R_2^D + M_1^{i''j''}}{P_2}$, and $c_1^{i''j''} + c_2^{i''j''} < c_2^{ij}$. Thus, given the liquidity strategy set σ , including the withdrawal strategies for late consumers $i' \neq i$, $\{\sigma^{i'j'}\}_{i' \in I, j' \in J}$, where $\sigma^{i'j'} = \{0, 0\}$, $\sigma^{ij} = \{0, 0\}$ is a weakly best response for all (λ, r_2) and $e \in E$ and a strictly best response for $(\lambda, r_2) > (\tilde{\lambda}, \tilde{r}_2)$ for all $e \in E$. Suppose instead there is any deviation in the interbank lending strategy $\sigma^{Bj''}$ by any bank j'' , such that $\sigma^{Bj} \neq L_1^{Bj}$ for $j \in \{h, l\}$. If $\sigma^{Bj''} \neq L_1^{Bj''}$, then $c_2^{ij''} < c_2^*$. Thus, given the liquidity strategy set σ , $\sigma^{Bj} = L_1^{Bj}$ is a strictly best response for all (λ, r_2) and $e \in E$. Hence, σ is a Nash equilibrium of the liquidity game with no bank runs.

Proof for Lemma 2. Consider a liquidity strategy set σ without interbank lending, $\sigma^{Bj} = 0$ for $j \in L$, and with a set of withdrawal strategies $\{\sigma^{ij}\}_{i \in I, j \in J}$, where each withdrawal strategy is a best response to σ : for each $i \in I$ and $j \in J$, for σ^{ij} and any feasible deviation $\sigma^{i''j''}$, $c_1^{ij} + c_2^{ij} \geq c_1^{i''j''} + c_2^{i''j''}$. Suppose there is a deviation by a liquid bank l'' : $L_1^{Bhl''} > 0$. There is a default by the borrowing bank h on the interbank loan at $t = 2$, $\delta_2^{Bhl''} < 1$. Hence, $c_2^{il''} < c_2^{il}$. Thus, $\sigma^{Bj} = 0$ is a strictly best response for all $j \in L$, and σ is a Nash equilibrium.

Proof for Proposition 4. The cutoff e_0 is implicitly defined by $c_2^{ih} = c_1^{ih} = c_1^h = c_1^l$, which

can be expressed as $\frac{\delta_2^h R_2^D}{P_2} = \frac{R_1^D}{P_1}$, or $\frac{P_2}{P_1} = \frac{\delta_2^h R_2^D}{R_1^D}$. For $e' \in (0, e_0]$, $c_2^{ih} \geq c_1^{ih} = c_1^h = c_1^l$. With $\frac{\delta_2^h R_2^D}{P_2} \geq \frac{R_1^D}{P_1}$, or $\frac{P_2}{P_1} \leq \frac{\delta_2^h R_2^D}{R_1^D}$, then $w^{ih} = 0$. However, $\delta_2^h < 1$, $a_0^h > a_0^*$, $P_1 < P_1^*$, $\frac{R_1^{Fj}}{P_1} > \frac{R_1^{Fj}}{P_1^*}$, and $c_2^{ih} < c_2^{il} < c_2^*$. The cutoff e_1 is implicitly defined by $c_2^{il} = c_1^{il} = c_1^l$, which can be expressed as $\frac{R_2^D}{P_2} = \frac{R_1^D}{P_1}$, or $\frac{P_2}{P_1} = \frac{R_2^D}{R_1^D}$. For $e'' \in (e_0, e_1]$, $0 = c_2^h < c_1^{ih} = c_1^h < c_1^* < c_1^l$. With $\frac{\delta_2^h R_2^D}{P_2} < \frac{R_1^D}{P_1}$, i.e. $\delta_2^h < \frac{R_1^D P_2}{R_2^D P_1}$, then $w^{ih} = 1$, which implies $\delta_1^h < 1$ and $\delta_2^h = 0$. For $e''' \in (e_1, 1 - \lambda)$, $c_2^{il} = c_1^{il} = c_1^l$. Suppose $w^{il} = 0$: then $c_2^{il} < c_1^{il} = c_1^l$, since $\frac{R_2^D}{P_2} < \frac{R_1^D}{P_1}$, i.e., $\frac{P_2}{P_1} > \frac{R_2^D}{R_1^D}$. Hence, $w^{il} > 0$, such that $c_2^{il} = c_1^{il} = c_1^l$, i.e. $\frac{R_2^D}{P_2} = \frac{R_1^D}{P_1}$, i.e., $\frac{P_2}{P_1} = \frac{R_2^D}{R_1^D}$.

Proof for Proposition 5. Consider a liquidity strategy set σ with complete early withdrawals, $w^{ij} = 1$ for all $i \in I, j \in J$, in the form of demands for digital currency. For the case of $M_0^B < D_0 R_2^D$, for each bank $j \in J$, it is not feasible to pay these early withdrawal demands in currency, which implies the bank defaults at $t = 1$. For the case of $M_0^B \geq D_0 R_2^D$, for each bank $j \in J$, the bank's budget constraint at $t = 1$ implies that the bank defaults at $t = 1$, $\delta_1^j < 1$, and hence $\delta_2^j = 0$. Suppose there is any deviation in the withdrawal strategy $\sigma^{i''j''}$ by any late consumer i'' at any bank j'' . For $w^{i''j''} < 1$, the late consumer receives no amount for the withdrawal of $(1 - w^{i''j''})$ at $t = 2$. For an early withdrawal demand not in digital currency, consumption $c_1^{i''j''} + c_2^{i''j''}$ is unchanged. Hence, σ is a Nash equilibrium.

Proof for Proposition 6. Consider any liquidity strategy set σ with an issuer lending strategy σ^{Yj} that lends to each bank $j \in J$ the amount of its liquidity shortfall from not receiving optimal interbank borrowing or incurring early withdrawals:

$$L_1^{BjY} = L_0^F R_1^F - L_1^{Fj} - \sum_{j' \in J} L_1^{Bj'j} + M_0^B - (\lambda^j + \sum_{i \in I} w^{ij}) D_0 R_1^D \quad \forall j \in J. \quad (4.1)$$

There is no default for any bank $j \in J$, which implies that any bank interbank lending strategy that does not provide optimal interbank lending, and any late consumer withdrawal strategy that has a positive amount of early withdrawals, is not a best response. Hence, with such an issuer lending strategy, the optimal Nash equilibrium is unique, in which the issuer does not lend, has zero consumption, and depositors have optimal consumption. Thus, this issuer lending strategy is chosen under a regime of fiat money or public digital currency. However, under a private digital currency regime, the issuer has positive consumption in an interbank freeze or withdrawal run equilibrium. Thus, the issuer does not choose the lending strategy to eliminate these financial crisis equilibria.